ADDENDUM TO
Curriculum Vitae of David W. K. Yeung

Mathematical Formulae Developed

One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers... ~Heinrich Hertz (1847-1894)

Part A: Control Theory

Theorem A1. (Random-horizon Bellman Equation)

A set of strategies \(\{u_k = \psi_k(x), \text{for } k \in T\}\) provides an optimal solution to the control Problem A1 (see below) if there exist functions \(V(k, x)\), for \(k \in T\), such that the following recursive relations are satisfied:

\[
V(T + 1, x) = q_{T+1}(x),
\]

\[
V(T, x) = \max_{u_T} \left\{ g_T(x, u_T) + V[T + 1, f_T(x, u_T)] \right\},
\]

\[
V(\tau, x) = \max_{u_\tau} \left\{ g_\tau(x, u_\tau) + \sum_{\tau' = \tau}^{T} \theta_{\tau'} q_{\tau'}[f_\tau(x, u_\tau)] \right\}, \quad \text{for } \tau \in \{1, 2, \ldots, T - 1\}. \quad \blacksquare
\]

Problem A1.

Consider the discrete-time dynamic programming problem with \(\hat{T}\) stages where \(\hat{T}\) is a random variable with range \(\{1, 2, \ldots, T\}\) and corresponding probabilities \(\{\theta_1, \theta_2, \ldots, \theta_T\}\). Conditional upon the reaching of stage \(\tau\), the probability of the game would last up to stages \(\tau, \tau + 1, \ldots, T\) becomes respectively

\[
\sum_{\tau' = \tau}^{T} \theta_{\tau'}, \quad \sum_{\tau' = \tau}^{T} \theta_{\tau'}, \ldots, \quad \sum_{\tau' = \tau}^{T} \theta_{\tau'}. \]

The single-stage payoff of the decision-maker at stage \(k \in \{1, 2, \ldots, T\}\) is \(g_k(x_k, u_k)\). When the problem ends after stage \(\hat{T}\), player \(i\) will receive a terminal payment \(q_{\hat{T}+1}(x_{\hat{T}+1})\) in stage \(\hat{T} + 1\).
The state space of the problem is $X \in \mathbb{R}^n$ and the state dynamics is characterized by the difference equation:

$$x_{k+1} = f_k (x_k, u_k),$$

for $k \in \{1,2,\cdots,T\}$ and $x_1 = x^0$,

where $u_k \in \mathbb{R}^m$ is the control vector at stage $k$ and $x_k \in X$ is the state.

The objective to be maximized is

$$E \left\{ \sum_{k=1}^{T} g_k (x_k, u_k) + q_{T+1} (x_{T+1}) \right\} = \sum_{\tau=1}^{T} \left\{ \sum_{k=1}^{T} g_k (x_k, u_k) + q_{T+1} (x_{T+1}) \right\}.$$


**Theorem A2 (Random-horizon Stochastic Bellman Equation under Uncertain Future Payoff Structures)**

A set of strategies $\{u_k^{(\sigma)} = \phi_k^{(\sigma)}(x), \text{ for } \sigma_k \in \{1,2,\cdots,\eta_k\} \text{ and } k \in \{1,2,\cdots,T\}\}$ provides an optimal solution to the stochastic control Problem A2 if there exist functions $V^{(\sigma)}(k,x)$, for $k \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$V^{(\sigma)}(T+1,x) = q_{T+1}(x),$$

$$V^{(\sigma)}(T,x) = \max_{u_T} E_{\theta_T} \left\{ g_T(x,u_T;\theta_T^{(\sigma)}) + V^{(\sigma)}(T+1,f_T(x,u_T) + \theta_T) \right\},$$

$$V^{(\sigma)}(T,x) = \max_{u_T} E_{\theta_T} \left\{ g_T(x,u_T;\theta_T^{(\sigma)}) + \frac{\theta_T^{(\sigma)}}{\sum_{\gamma} \theta_T^{(\gamma)}} q_{T+1}[f_T(x,u_T) + \theta_T] \right\} + \frac{\theta_T^{(\sigma)}}{\sum_{\gamma} \theta_T^{(\gamma)}} \sum_{\tau=1}^{T} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \mathcal{P}_{\tau+1}^{(\sigma)} V^{(\sigma_{\tau+1})} [\tau + 1, f_T(x,u_T) + \theta_T] \right\},$$

$$= E_{\theta_T} \left\{ g_T(x,u_T;\theta_T^{(\sigma)}) + \frac{\theta_T^{(\sigma)}}{\sum_{\gamma} \theta_T^{(\gamma)}} q_{T+1}[f_T(x,u_T) + \theta_T] \right\} + \frac{\theta_T^{(\sigma)}}{\sum_{\gamma} \theta_T^{(\gamma)}} \sum_{\tau=1}^{T} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \mathcal{P}_{\tau+1}^{(\sigma)} V^{(\sigma_{\tau+1})} [\tau + 1, f_T(x,u_T) + \theta_T] \right\},$$

for $\tau \in \{1,2,\cdots,T-1\}$.
Problem A2.
Consider the discrete time $\hat{T}$–stage stochastic dynamic optimization problem where $\hat{T}$ is a random variable with range $\{1,2,\ldots,T\}$ and corresponding probabilities \{\(\sigma_1, \sigma_2, \ldots, \sigma_T\)\}. Conditional upon the reaching of stage $\tau$, the probability of the game would last up to stages $\tau, \tau+1, \ldots, T$ becomes respectively
\[
\frac{\sigma_{\tau}}{T}, \frac{\sigma_{\tau+1}}{T}, \ldots, \frac{\sigma_{T}}{T}.
\]

The state space of the game is $X \in \mathbb{R}^m$ and the state dynamics of the game is characterized by the stochastic difference equation:
\[
x_{k+1} = f_k(x_k, u_k) + \partial_k,
\]
for $k \in \{1,2,\ldots,T\}$ and $x_1 = x^0$,
where $u_k \in U \subset \mathbb{R}^m$ is the control vector at stage $k$, $x_k \in X$ is the state, and $\partial_k$ is a sequence of statistically independent random variables.

The single-stage payoff at stage $k$ is $g_k(x_k, u_k; \theta_k)$ which is affected by a random variable $\theta_k$. In particular, $\theta_k$ for $k \in \{1,2,\ldots,T\}$ are independent random variables with range $\{\theta^1_k, \theta^2_k, \ldots, \theta^n_k\}$ and corresponding probabilities \{\(\lambda^1_k, \lambda^2_k, \ldots, \lambda^n_k\)\}. In stage 1, it is known that $\theta_1$ equals $\theta^1$ with probability $\lambda^1_1 = 1$. When the game ends after stage $\hat{T}$, a terminal payment $q_{\hat{T}+1}(x_{\hat{T}+1})$ will be given in stage $\hat{T}+1$.

The objective to be maximized is
\[
E_{\theta_1, \theta_2, \ldots, \theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{j=1}^{T} \sigma_j \left[ \sum_{k=1}^{T} g'_k(x_k, u_k; \theta_k) + q(x_{T+1}) \right] \right\},
\]
where $E_{\theta_1, \theta_2, \ldots, \theta_1, \theta_2, \ldots, \theta_T}$ is the expectation operation with respect to the random variables $\theta_1, \theta_2, \ldots, \theta_T$ and $\theta_1, \theta_2, \ldots, \theta_T$. Since there is no uncertainty in the payoff structure in stage $T+1$, we denote $\sigma_{T+1} = 1$, $\theta^0_{T+1} = \theta^1_{T+1}$ with probability $\lambda^0_{T+1} = \lambda^1_{T+1} = 1$ for notational convenience. The payoffs of the players are transferable.


Part B: Game Theory

Theorem B1. (Random-horizon (Hamilton-Jacobi-Bellman) HJB Equations)
A set of strategies $\{\phi'_i(x)\}$, for $k \in T$ and $i \in N$ provides a feedback Nash equilibrium solution to the game Problem B1 (see below) if there exist functions $V'(k,x)$, for $k \in T$ and $i \in N$, such that the following recursive relations are satisfied:
\[
V'(T, x) = \max_{u_i} \left\{ g_i^i [x, \phi_i^i(x), \phi_i^1(x), \phi_i^{n-1}(x), u_i', \phi_i^{n+1}(x), \ldots, \phi_i^n(x)] + q_{T-1}^i [f_i^i (x, u_i')] \right\},
\]
\[
V'(\tau, x) = \max_{u_i} \left\{ g_i^i [x, \phi_i^i(x), \phi_i^1(x), \phi_i^{n-1}(x), u_i', \phi_i^{n+1}(x), \ldots, \phi_i^n(x)] + \sum_{\tau = \tau}^{\theta_{\tau+1}} q_{T+1}^i [f_i^i (x, u_i')] \right\},
\]
\[
V'[\tau + 1, f_k (x, \phi_k^i(x), \phi_k^1(x), \phi_k^{n-1}(x), u_k', \phi_k^{n+1}(x), \ldots, \phi_k^n(x))] \right\},
\]
for \( \tau \in \{1, 2, \ldots, T-1\} \).

**Problem B1.**

Consider the \( n \)-person dynamic game with \( T \) stages where \( T \) is a random variable with range \( \{1, 2, \ldots, T\} \) and corresponding probabilities \( \{\theta_1, \theta_2, \ldots, \theta_T\} \). Conditional upon the reaching of stage \( \tau \), the probability of the game would last up to stages \( \tau, \tau + 1, \ldots, T \) becomes respectively

\[
\frac{\theta_{\tau}}{\sum_{\tau = \tau}^{\theta_{\tau+1}}} = \frac{\theta_{\tau+1}}{\sum_{\tau = \tau}^{\theta_{\tau+1}}} = \frac{\theta_T}{\sum_{\tau = \tau}^{\theta_T}}.
\]

The payoff of player \( i \) at stage \( k \in \{1, 2, \ldots, T\} \) is \( g_k^i [x_k, u_k^1, u_k^2, \ldots, u_k^n] \). When the game ends after stage \( T \), player \( i \) will receive a terminal payment \( q_{T+1}^i (x_{T+1}) \) in stage \( T+1 \).

The state space of the game is \( X \in R^n \) and the state dynamics of the game is characterized by the difference equation:

\[
x_{k+1} = f_k (x_k, u_k^1, u_k^2, \ldots, u_k^n),
\]

for \( k \in \{1, 2, \ldots, T\} \equiv T \) and \( x_1 = x^0 \),

where \( u_i' \in R^n \) is the control vector of player \( i \) at stage \( k \) and \( x_k \in X \) is the state.

The objective of player \( i \) is

\[
E \left\{ \sum_{k=1}^{T} g_k^i [x_k, u_k^1, u_k^2, \ldots, u_k^n] + q_{T+1}^i (x_{T+1}) \right\}
\]

\[
= \sum_{T=1}^{\theta_T} \theta_{T} \left\{ \sum_{k=1}^{T} g_k^i [x_k, u_k^1, u_k^2, \ldots, u_k^n] + q_{T+1}^i (x_{T+1}) \right\},
\]

for \( i \in \{1, 2, \ldots, n\} \equiv N \).

Theorem B2. (Random-horizon Stochastic HJB Equations under Uncertain Future Payoff Structures)

A set of strategies \(\{u^i_\tau = \phi^i_{(\sigma_j)_\tau}(x), \text{ for } \sigma_j \in \{1, 2, \ldots, \eta_j\}, \tau \in \{1, 2, \ldots, T\}\) and \(i \in N\) constitutes a Nash equilibrium solution to the game Problem B2 (see below) if there exist functions \(V^{(\sigma_j)_\tau}(x, \tau), \text{ for } \sigma_j \in \{1, 2, \ldots, \eta_j\}, \tau \in \{1, 2, \ldots, T\}\) and \(i \in N\), such that the following recursive relations are satisfied:

\[
V^{(\sigma_j)_\tau}(T + 1, x) = q_{T+1}(x),
\]

\[
V^{(\sigma_j)_\tau}(T, x) = \max_{u^i_\tau} E_{\theta^i_\tau} \left\{ g^i_T\left[ x, \phi^i_{(\sigma_j)_T}(x), \phi^i_{(\sigma_j)_{T+1}}(x), \ldots, \phi^i_{(\sigma_j)_{T+n}}(x), u^i_{(\sigma_j)_T}, \phi^i_{(\sigma_j)_{T+1}}(x), \ldots \right] \\
+ \phi^i_{(\sigma_j)_\tau}(x) ; \theta^i_\tau \right\} + V^{(\sigma_j)_{T+1}}(T + 1, f^i_T(x, \phi^i_{(\sigma_j)^{(r)}}(x)) + \theta^i_\tau),
\]

\[
V^{(\sigma_j)_\tau}(\tau, x) = \max_{u^i_\tau} E_{\theta^i_\tau} \left\{ g^i_T\left[ x, \phi^i_{(\sigma_j)_T}(x), \phi^i_{(\sigma_j)_{T+1}}(x), \ldots, \phi^i_{(\sigma_j)_{T+n}}(x), u^i_{(\sigma_j)_T}, \phi^i_{(\sigma_j)_{T+1}}(x), \ldots \right] \\
+ \frac{\sigma_\tau}{\sigma_\tau} q_{T+1}[f^i_T(x, \phi^i_{(\sigma_j)^{(r)}}(x)) + \theta^i_\tau] \right\}, \tau \in \{1, 2, \ldots, T - 1\};
\]

for \(\sigma_j \in \{1, 2, \ldots, \eta_j\}, \tau \in \{1, 2, \ldots, T\}\) and \(i \in N\); where

\[
\phi^i_{(\sigma_j)^{(r)}}(x) = f^i_{\tau}[x, \phi^i_{(\sigma_j)_T}(x), \phi^i_{(\sigma_j)_{T+1}}(x), \ldots, \phi^i_{(\sigma_j)_{T+n}}(x), u^i_{(\sigma_j)_T}, \phi^i_{(\sigma_j)_{T+1}}(x), \ldots, \phi^i_{(\sigma_j)^{(r)}}(x)];
\]

for \(\tau \in \{1, 2, \ldots, T\}\). \(\blacksquare\)

**Problem B2**

Consider the discrete time \(\hat{T}\) - stage stochastic dynamic game problem where \(\hat{T}\) is a random variable with range \(\{1, 2, \ldots, T\}\) and corresponding probabilities \(\{\sigma_1, \sigma_2, \ldots, \sigma_T\}\). Conditional upon the reaching of stage \(\tau\), the probability of the game would last up to stages \(\tau, \tau + 1, \ldots, T\) becomes respectively

\[
\sigma_\tau^{\hat{T}}, \sigma_{\tau+1}^{\hat{T}}, \ldots, \sigma_T^{\hat{T}}.
\]

The state space of the game is \(X \in R^m\) and the state dynamics of the game is characterized by the stochastic difference equation:

\[
x_{\tau+1} = f_{\tau}(x_{\tau}, u^1_{\tau}, u^2_{\tau}, \ldots, u^m_{\tau}) + \theta_{\tau},
\]

for \(k \in \{1, 2, \ldots, T\}\) and \(x_0 = x^0\), where \(u^i_{\tau} \in U^i \subset R^m\) is the control vector of player \(i\) at stage \(k\), \(x_\tau \in X\) is the state, and \(\theta_{\tau}\) is a sequence of statistically independent random variables.
The payoff of player $i$ at stage $k$ is $g_k^i[x_k, u_k^1, u_k^2, \ldots, u_k^n; \theta_k]$ which is affected by a random variable $\theta_k$. In particular, $\theta_k$ for $k \in \{1, 2, \ldots, T\}$ are independent random variables with range $\{\theta_k^1, \theta_k^2, \ldots, \theta_k^n\}$ and corresponding probabilities $\{\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^n\}$.

In stage 1, it is known that $\theta_1$ equals $\theta_1^i$ with probability $\lambda_1^i = 1$. When the game ends after stage $\hat{T}$, a terminal payment $q^i_{\hat{T}+1}(x_{\hat{T}+1})$ will be given to player $i$ in stage $\hat{T} + 1$.

The objective that player $i$ seeks to maximize is

$$
E_{\theta_1, \theta_2, \ldots, \theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{T=1}^{T} \sigma_i \left[ \sum_{k=1}^{T} g_k^i[x_k, u_k^1, u_k^2, \ldots, u_k^n; \theta_k] + q^i(x_{T+1}) \right] \right\},
$$

for $i \in \{1, 2, \ldots, n\} = N$, where $E_{\theta_1, \theta_2, \ldots, \theta_1, \theta_2, \ldots, \theta_T}$ is the expectation operation with respect to the random variables $\theta_1, \theta_2, \ldots, \theta_T$ and $\theta_1^1, \theta_2^1, \ldots, \theta_T^1$. Since there is no uncertainty in the payoff structure in stage $T+1$, we denote $\sigma_{T+1} = 1$, $\theta_{T+1}^i = \theta_{T+1}^i$ with probability $\lambda_{T+1}^i = \lambda_{T+1}^i = 1$ for notational convenience.


**Theorem B3. (Nontransferable Individual Payoff in Stochastic Dynamic Cooperation)**

If there exists a set of controls $\{u_k^i = \psi_k^{(a)}(x)\}$, for $k \in \kappa \equiv \{1, 2, \ldots, T\}$ and $i \in N$ and value functions $\{W^{(a)}(k, x)\}$, for $k \in \kappa$ which provide an optimal solution to the stochastic control Problem B3 (see below) then the individual player’s payoff $W^{(a)i}(t, x)$ for $i \in N$ and $k \in \kappa$ satisfy the following recursive relations:

$$
W^{(a)}(T+1, x) = \sum_{j=1}^{n} \alpha^j q^j(x_{T+1}),
$$

$$
W^{(a)}(k, x) = \max_{u_k^1, u_k^2, \ldots, u_k^n} E_{\theta_k} \left\{ \sum_{j=1}^{n} \alpha^j g_k^i(x_k, u_k^1, u_k^2, \ldots, u_k^n) + W^{(a)}(k+1, f_k(x_k, u_k^1, u_k^2, \ldots, u_k^n) + \theta_k) \right\},
$$

$$
W^{(a)i}(T+1, x) = q^i(x_{T+1}),
$$

$$
W^{(a)}(k, x) = E_{\theta_k} \left\{ g_k^i[x_k, \psi_k^{(a)(1)}(x), \psi_k^{(a)(2)}(x), \ldots, \psi_k^{(a)n}(x)] + W^{(a)i}(k+1, f_k(x_k, \psi_k^{(a)(1)}(x), \psi_k^{(a)(2)}(x), \ldots, \psi_k^{(a)n}(x)) + \theta_k) \right\}, \text{ for } i \in N \text{ and } k \in \kappa.
$$

**Problem B3.**
Consider the problem of maximizing the joint weighted expected payoff of the players in a cooperative stochastic dynamic game with non-transferrable payoffs which maximizes
\[
E_{\theta_1, \theta_2, \ldots, \theta_n} \left\{ \sum_{j=1}^{n} \left[ \sum_{k=1}^{T} \alpha^j g^j_k(x_k, u^1_k, u^2_k, \ldots, u^n_k) + \alpha^j q^j(x_{T+1}) \right] \right\}
\]
subject to
\[
x_{k+1} = f_k(x_k, u^1_k, u^2_k, \ldots, u^n_k) + \theta_k, \quad \text{for } k \in \{1, 2, \ldots, T\} \equiv \kappa \quad \text{and} \quad x_1 = x^0_1,
\]
where \(u^i_k \in \mathbb{R}^m\) is the control vector of player \(i\) at stage \(k\), \(x_k \in X \subset \mathbb{R}^m\) is the state, and \(\theta_k\) is a set of statistically independent random variables, and \(\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)\) for \(\sum_{j=1}^{n} \alpha^j = 1\) is a set agreed-upon weights.


**Theorem B4. (Nontransferable Individual Payoff in Continuous-time Stochastic Dynamic Cooperation)**

If there exists a set of controls \(\{u^i_{\alpha}(t) = \psi^i_{\alpha}(t, x)\), for \(i \in N\) \} and value functions \(W^i_{\alpha}(t, x) : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}\) which provide an optimal solution to the stochastic control Problem B4 (see below), then the individual player’s payoff \(W^i_{\alpha}(t, x) : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}\) for \(i \in N\) satisfy the following partial differential equations:
\[
-W^i_{\alpha}(t, x) - \frac{1}{2} \sum_{h=1}^{n} \Omega^h_{\kappa}(t, x)W^i_{\alpha}(t, x) = \max_{u_1, u_2} \left\{ \left[ \sum_{j=1}^{n} \alpha^j g^j(t, x, u_1, u_2, \ldots, u_n) \right] \exp \left[ -\int_{t_0}^{t} r(y)dy \right] + W^i_{\alpha}(t, x) f(t, x, u_1, u_2, \ldots, u_n) \right\},
\]
where
\[
W^i_{\alpha}(T, x) = \exp \left[ -r(T - t_0) \right] \sum_{j=1}^{n} \alpha^j q^j(x),
\]
\[
-W^i_{\alpha}(t, x) - \frac{1}{2} \sum_{h=1}^{n} \Omega^h_{\kappa}(t, x)W^i_{\alpha}(t, x) = g^i(t, x, \psi^1_{\alpha}(t, x), \psi^2_{\alpha}(t, x), \ldots, \psi^n_{\alpha}(t, x)) \exp \left[ -\int_{t_0}^{t} r(y)dy \right] + W^i_{\alpha}(t, x) f(t, x, \psi^1_{\alpha}(t, x), \psi^2_{\alpha}(t, x), \ldots, \psi^n_{\alpha}(t, x)) \quad \text{and}
\]
\[
W^i_{\alpha}(T, x) = \exp \left[ -\int_{t_0}^{T} r(y)dy \right] q^i(x), \quad \text{for } i \in N.
\]
Consider the problem of maximizing the joint weighted expected payoff of the players in a cooperative stochastic differential game with non-transferrable payoffs which maximizes

$$E_0 \left\{ \sum_{j=1}^{n} \int_{t_0}^{T} \alpha^j \, g^j [s, x(s), u_1(s), u_2(s), \ldots, u_n(s)] \exp \left[ - \int_{t_0}^{s} \, r(y) \, dy \right] ds 
+ \sum_{j=1}^{n} \exp \left[ - \int_{t_0}^{T} \, r(y) \, dy \right] \alpha^j \, q^j (x(T)) \right\},$$

subject to

$$dx(s) = f [s, x(s), u_1(s), u_2(s), \ldots, u_n(s)] ds + \sigma [s, x(s)] dz(s), \quad x(t_0) = x_0,$$

where $\sigma [s, x(s)]$ is a $n \times \Theta$ matrix and $z(s)$ is a $\Theta$-dimensional Wiener process and $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ for $\sum_{j=1}^{n} \alpha^j = 1$ is a set agreed-upon weights.

Let $\Omega [s, x(s)] = \sigma [s, x(s)] \sigma^T [s, x(s)]$ denote the covariance matrix with its element in row $h$ and column $\zeta$ denoted by $\Omega[h\zeta][s, x(s)]$. $u_i \in U_i \subset \text{comp}R^i$ is the control vector of player $i$, for $i \in \{1, 2\}$.

**References:**


**Theorem B5.** (Subgame-consistent Payoff Distribution Procedure (PDP) for Discrete-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic dynamic game Problem B5 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^i(k, x^*_k)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x^*_k\}_{k=0}^{\infty}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B'_k(x^*_k) = (1 + r)^{k-1} \left\{ \xi^i(k, x^*_k) \right\}
- E_{\theta_k} \left\{ \xi^i[k + 1, f(x^*_k, \psi^i_1(x^*_k), \psi^2_2(x^*_k), \ldots, \psi^n_0(x^*_k)) + \theta_k] \right\},$$

for $i \in N$,
given to player $i$ at stage $k \in \kappa$, if $x^*_k \in X^*_k$ would lead to the realization of the imputation $\{ \xi^i(k, x^*_k) \}$, for $i \in N$ and $k \in \kappa$;

where
\{\psi^i_k(x), \text{for } k \in \kappa \text{ and } i \in N\} \text{ is a set of strategies that provides an optimal solution to the Problem B5 yielding functions } W(k,x), \text{ for } k \in K, \text{ such that the following recursive relations are satisfied:}

\[
W(k,x) = \max_{u_1, u_2, \ldots, u_k} E_\theta \left\{ \sum_{j=1}^n g^i_j [x, u^1_k, u^2_k, \ldots, u^n_k] \left( \frac{1}{1+r} \right)^{k-1} \right.
+ W[k+1, f_k (x, u^1_k, u^2_k, \ldots, u^n_k) + \theta_k] \left. \right\},
\]

\[
W(T+1,x) = \sum_{j=1}^n q^j_{T+1}(x) \left( \frac{1}{1+r} \right)^T.
\]

**Problem B5.**
Consider the general \textit{T} – stage \textit{n} – person discrete-time cooperative stochastic dynamic game with initial state \(x^0\). The state space of the game is \(X \in \mathbb{R}^m\) and the state dynamics of the game is characterized by the stochastic difference equation:

\[
x_{k+1} = f_k (x_k, u^1_k, u^2_k, \ldots, u^n_k) + \theta_k,
\]

for \(k \in \{1,2,\ldots,T\} \equiv \kappa \) and \(x_1 = x^0\),

where \(u^i_k \in \mathbb{R}^n\) is the control vector of player \(i\) at stage \(k\), \(x_k \in X\) is the state, and \(\theta_k\) is a set of statistically independent random variables.

The objective of player \(i\) is

\[
E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{i=1}^T g^i_{\xi} [x^\xi_k, u^1_k, u^2_k, \ldots, u^n_k] \left( \frac{1}{1+r} \right)^{t-1} + q^j_{T+1}(x_{T+1}) \left( \frac{1}{1+r} \right)^T \right\},
\]

for \(i \in \{1,2,\ldots,n\} \equiv N\),

where \(r\) is the discount rate and \(E_{\theta_1, \theta_2, \ldots, \theta_T}\) is the expectation operation with respect to the statistics of \(\theta_1, \theta_2, \ldots, \theta_T\).

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation \(\xi^i(k,x^*_k)\) for player \(i \in N\) in stage \(k \in \kappa\) along the cooperative trajectory \(\{x^*_k\}_{k=1}^T\). Examples of the imputation \(\xi^i(k,x^*_k)\) include:

(i) Sharing the extra gain from cooperation equally, and the imputation to player \(i\) becomes:

\[
\xi^i(k,x^*_k) = V^i(k,x^*_k) + \frac{1}{n} \left[ W(k,x^*_k) - \sum_{j=1}^n V^j(k,x^*_k) \right], \quad \text{for } i \in N \text{ and } k \in \kappa,
\]

where \(V^i(k,x^*_k)\) is the expected noncooperative payoff of player \(i\) and \(W(k,x^*_k)\) is the expected total cooperative payoff.

(ii) Share the total cooperative proportional to the players’ noncooperative payoffs, and the imputation to player \(i\) becomes:

\[
\xi^i(k,x^*_k) = \frac{V^i(k,x^*_k)}{\sum_{j=1}^n V^j(k,x^*_k)} \cdot W(k,x^*_k), \quad \text{for } i \in N \text{ and } k \in \kappa.
\]
Theorem B6. (Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic differential game Problem B6 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation \( \xi^{i(j)}(s, x^*_i) \) in current value at time \( s \) for player \( i \in N \) in time \( s \in [0, T] \) along the cooperative trajectory \( \{x_s^{i}\}_{s=0}^T \). A Payoff Distribution Procedure (PDP) with a payment equaling

\[
B_i(s, x^*_i) = - \left[ \xi^{i(j)}(m, x^*_i) \right]_{t=s} - \left[ \xi^{i(j)}(m, x^*_i) \right]_{t=s} f[s, x^*_i, y_i^*(s), y^*_2(s), \ldots, y_n^*(s, x^*_i)]
\]

\[
- \frac{1}{2} \sum_{h_x=1}^{m} \Omega^h_x(s, x^*_i) \left[ \xi^{i(j)}(m, x^*_i) \right]_{t=s}, \quad \text{for } i \in N \text{ and } x^*_i \in X^*_i,
\]

given to player \( i \) at time \( s \in [t_0, T] \) would lead to the realization of the imputation \( \{ \xi^{i(j)}(s, x^*_i) \}, \text{ for } i \in N \text{ and } s \in [t_0, T] \); where

\( \{ y^*_i(s, x) \}, \text{ for } i \in N \text{ and } s \in [t_0, T] \) is a set of strategies that provides an optimal solution to the Problem B6 yielding functions continuously twice differentiable functions \( W(t, x) : [t_0, T] \times R^m \rightarrow R \), which satisfy the following partial differential equation:

\[
-W_i(t, x) - \frac{1}{2} \sum_{h_x=1}^{m} \Omega^h_x(t, x) W_{x^*_i}(t, x) = \max_{u_1, u_2, \ldots, u_n} \left\{ \sum_{j=1}^{n} g^i[t, x, u_1, u_2, \ldots, u_n] \exp \left[ - \int_{t_0}^{t} r(y) dy \right] + W_i(t, x) \right\}, \quad \text{and}
\]

\[
W(T, x) = \sum_{j=1}^{n} g^i(x) \exp \left[ - \int_{t_0}^{T} r(y) dy \right].
\]

Problem B6.
Consider the \( n \)-person cooperative stochastic differential games in which player \( i \) seeks to maximize its expected payoffs:

\[
E_i \left\{ \int_{t_0}^{T} g^i[s, x(s), u_i(s), u_2(s), \ldots, u_n(s)] \exp \left[ - \int_{t_0}^{t} r(y) dy \right] ds \right\}
\]

\[
\]
$E_x \{ \cdot \}$ denoting the expectation operation taken at time $t_0$, and the dynamics of the state is

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + \sigma[s, x(s)]dz(s), \quad x(t_0) = x_0,$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a $\Theta$-dimensional Wiener process and the initial state $x_0$ is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)] \sigma[s, x(s)]'$ denote the covariance matrix with its element in row $h$ and column $\zeta$ denoted by $\Omega_{\zeta h}[s, x(s)]$. Moreover, $E[dx_\sigma] = 0$ and $E[dx_\sigma dt] = 0$ and $E[(dx_\sigma)^2] = dt$, for $\sigma \in [1, 2, \cdots, \Theta]$; and $E[dx_\sigma dx_\omega] = 0$, for $\sigma, \omega \in [1, 2, \cdots, \Theta]$ and $\sigma \neq \omega$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(V)}(s, x^*_i)$ in current value at time $s$ for player $i \in N$ in time $s \in [t_0, T]$ along the cooperative trajectory $\{x^*_s\}_{s=t_0}^T$.

**References:**


**Theorem B7. (Subgame-consistent PDP for Random-horizon Dynamic Cooperation)**

Consider the random-horizon cooperative dynamic game Problem 9 (see below) in which the players agree to maximize their joint payoff and share the cooperative gain according to the imputation $\xi^{(V)}(\tau, x^*_i)$ for player $i \in N$ in stage $\tau \in \mathcal{K}$ along the cooperative trajectory $\{x^*_s\}_{s=1}^T$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_i^*(x^*_i) = \xi^{(V)}(\tau, x^*_i) = \sum_{\tau=1}^{T} \theta^V_{\tau} \xi^{(V)}(\tau+1, f_i [x_\tau, \psi^1_\tau(x_\tau), \psi^2_\tau(x_\tau), \cdots, \psi^n_\tau(x_\tau)])$$

$$- \frac{\theta^V_{\tau+1}}{\sum_{\tau=1}^{T} \theta^V_{\tau+1}} q^{(V)}_\tau [f_i [x_\tau, \psi^1_\tau(x_\tau), \psi^2_\tau(x_\tau), \cdots, \psi^n_\tau(x_\tau)]]$$

for $i \in N$,

given to player $i$ at stage $\tau \in \mathcal{K}$ would lead to the realization of the imputation $\xi^{(V)}(\tau, x^*_i)$ for player $i \in N$ in stage $\tau \in \mathcal{K}$; where

$\{\psi^i_\tau(x), \text{ for } \tau \in \mathcal{K} \text{ and } i \in N\}$ is a set of strategies that provides a group optimal solution to the Problem 9 yielding functions $W(k, x), \text{ for } \tau \in \mathcal{K}$, such that the following recursive relations are satisfied:
\[ W(T+1, x) = \sum_{j=1}^{n} q_{T+1}^j(x), \]
\[ W(T, x) = \max_{u_1^j, u_2^j, \ldots, u_T^j} \left\{ \sum_{j=1}^{n} g_f^j(x, u_1^j, u_2^j, \ldots, u_T^j) + q_{T+1}^j(f(x, u_1^j, u_2^j, \ldots, u_T^j)) \right\}, \]
\[ W(\tau, x) = \max_{u_1^j, u_2^j, \ldots, u_T^j} \left\{ \sum_{j=1}^{n} g_f^j(x, u_1^j, u_2^j, \ldots, u_T^j) + \frac{\theta}{T} \sum_{\tau = \tau}^T q_{\tau+1}^j(f(x, u_1^j, u_2^j, \ldots, u_T^j)) \right\} + \frac{\sum_{\tau = \tau}^T \theta}{T}, \]
for \( \tau \in \{1, 2, \ldots, T-1\}. \)

**Problem B7.**
Consider the \( n \)-person cooperative dynamic game with \( \hat{T} \) stages where \( \hat{T} \) is a random variable with range \( \{1, 2, \ldots, T\} \) and corresponding probabilities \( \{\theta_1, \theta_2, \ldots, \theta_T\} \). Conditional upon the reaching of stage \( \tau \), the probability of the game would last up to stages \( \tau, \tau + 1, \ldots, T \) becomes respectively

\[ \sum_{\tau = \tau}^T \theta_\tau, \sum_{\tau = \tau}^T \theta_\tau, \ldots, \sum_{\tau = \tau}^T \theta_\tau. \]

The payoff of player \( i \) at stage \( k \in \{1, 2, \ldots, T\} \) is \( g_f^i(x, u_1^k, u_2^k, \ldots, u_T^k) \). When the game ends after stage \( \hat{T} \), player \( i \) will receive a terminal payment \( q_{\hat{T}+1}^i(x_{\hat{T}+1}) \) in stage \( \hat{T}+1 \).

The state space of the game is \( X \in R^m \) and the state dynamics of the game is characterized by the difference equation:

\[ x_{k+1} = f_k(x_k, u_1^k, u_2^k, \ldots, u_T^k), \]
for \( k \in \{1, 2, \ldots, T\} \equiv \kappa \) and \( x_0 = x^0 \),

where \( u_1^k \in R^m \) is the control vector of player \( i \) at stage \( k \) and \( x_k \in X \) is the state.

The objective of player \( i \) is

\[ E \left\{ \sum_{k=1}^{\hat{T}} g_f^i(x, u_1^k, u_2^k, \ldots, u_T^k) + q_{\hat{T}+1}^i(x_{\hat{T}+1}) \right\} \]
\[ = \sum_{\tau = 1}^{\hat{T}} \theta_{\tau} \left\{ \sum_{k=1}^{\tau} g_f^i(x, u_1^k, u_2^k, \ldots, u_T^k) + q_{\tau+1}^i(x_{\tau+1}) \right\}, \]
for \( i \in \{1, 2, \ldots, n\} \equiv N \).

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation \( \zeta_i(k, x_k) \) for player \( i \in N \) in stage \( k \in \kappa \) along the cooperative trajectory \( \{x_k\}_{k=1}^{\hat{T}} \).
Theorem B8. (Subgame-consistent PDP for Discrete-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the randomly furcating cooperative stochastic dynamic game Problem B8 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(\sigma_I)}(k, x_k^*) = [\xi^{(\sigma_I,1)}(k, x_k^*), \xi^{(\sigma_I,2)}(k, x_k^*), \cdots, \xi^{(\sigma_I,|n|)}(k, x_k^*)]$ along the cooperative trajectory given that $\theta_k^{\sigma_I}$ has occurred in stage $k$, for $\sigma_I \in \{1, 2, \cdots, \eta_I\}$ and $k \in \{1, 2, \cdots, T\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$
B_k^{(\sigma_I)}(x_k^*) = \xi^{(\sigma_I)}(k, x_k^*)
$$

for $i \in N$, given to player $i$ at stage $k \in \{1, 2, \cdots, T\}$, if $\theta_k^{\sigma_i}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation $\xi^{(\sigma_I)}(k, x_k^*)$ for $k \in \{1, 2, \cdots, T\}$; where

where $\psi_i^{(\sigma_I)}(x) = \{\psi_i^{(\sigma_I,1)}(x), \psi_i^{(\sigma_I,2)}(x), \cdots, \psi_i^{(\sigma_I,|n|)}(x)\}$, for $\sigma_I \in \{1, 2, \cdots, \eta_I\}$ and $t \in \{1, 2, \cdots, T\}$ is a set of strategies that provides a group optimal solution to Problem B8 yielding value functions $W^{(\sigma_I)}(t, x)$, for $\sigma_I \in \{1, 2, \cdots, \eta_I\}$ and $t \in \{1, 2, \cdots, T\}$, such that the following recursive relations are satisfied:

$$
W^{(\sigma_I)}(T + 1, x) = \sum_{j=1}^{n} q_j^{(x)}(x),
$$

$$
W^{(\sigma_I)}(T, x) = \max_{u_I^{(\sigma_I,1)}, u_I^{(\sigma_I,2)}, \cdots, u_I^{(\sigma_I,n)}} \mathbb{E}_{\theta_T} \left\{ \sum_{j=1}^{n} g_j^{(x, u_T^{(\sigma_I,1)}, u_T^{(\sigma_I,2)}, \cdots, u_T^{(\sigma_I,n)}, \theta_T^{\sigma_I}} + W^{(\sigma_I,1)}[T + 1, f_T(x, u_T^{(\sigma_I,1)}, u_T^{(\sigma_I,2)}, \cdots, u_T^{(\sigma_I,n)}, \theta_T^{\sigma_I})] \right\},
$$

$$
W^{(\sigma_I)}(t, x) = \max_{u_I^{(\sigma_I,1)}, u_I^{(\sigma_I,2)}, \cdots, u_I^{(\sigma_I,n)}} \mathbb{E}_{\theta_T} \left\{ \sum_{j=1}^{n} g_j^{(x, u_t^{(\sigma_I,1)}, u_t^{(\sigma_I,2)}, \cdots, u_t^{(\sigma_I,n)}, \theta_t^{\sigma_I})} + \sum_{\sigma_{r+1}^{\sigma_I=1}}^{\eta_I=1} W^{(\sigma_{r+1})}[t + 1, f_T(x, u_t^{(\sigma_{r+1}),1}, u_t^{(\sigma_{r+1}),2}, \cdots, u_t^{(\sigma_{r+1}),n}) + \theta_t^{\sigma_I}] \right\},
$$

for $\sigma_I \in \{1, 2, \cdots, \eta_I\}$ and $t \in \{1, 2, \cdots, T - 1\}$. ■

Problem B8.

Consider the $T$–stage $n$–person randomly furcating cooperative stochastic dynamic game with initial state $x^0$. The state space of the game is $X \in \mathbb{R}^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k(x_k, u_{k}^{1}, u_{k}^{2}, \ldots, u_{k}^{n}) + \theta_k,$$

for $k \in \{1, 2, \ldots, T\}$ and $x_1 = x^0$.

where $u_{k}^{i} \in U_{k}^{i} \subset \mathbb{R}^{m}$ is the control vector of player $i$ at stage $k$, $x_k \in X$ is the state, and $\theta_k$ is a sequence of statistically independent random variables.

The payoff of player $i$ at stage $k$ is $g_{k}^{i}(x_{k}, u_{k}^{1}, u_{k}^{2}, \ldots, u_{k}^{n}; \theta_k)$ which is affected by a random variable $\theta_k$. In particular, $\theta_k$ for $k \in \{1, 2, \ldots, T\}$ are independent discrete random variables with range $\{\theta_{k}^{1}, \theta_{k}^{2}, \ldots, \theta_{k}^{n}\}$ and corresponding probabilities $\{\lambda_{1}^{k}, \lambda_{2}^{k}, \ldots, \lambda_{n}^{k}\}$, where $\eta_k$ is a positive integer for $k \in \{1, 2, \ldots, T\}$. In stage 1, it is known that $\theta_1$ equals $\theta_1$ with probability $\lambda_1^1 = 1$.

The objective that player $i$ seeks to maximize is

$$E_{\theta_1, \theta_2, \ldots, \theta_n} \left\{ \sum_{k=1}^{T} g_{k}^{i}(x_{k}, u_{k}^{1}, u_{k}^{2}, \ldots, u_{k}^{n}; \theta_k) + q^i(x_{T+1}) \right\},$$

for $i \in \{1, 2, \ldots, n\} \equiv N$,

where $E_{\theta_1, \theta_2, \ldots, \theta_n}$ is the expectation operation with respect to the random variables $\theta_1, \theta_2, \ldots, \theta_n$ and $q^i(x_{T+1})$ is a terminal payment given at stage $T+1$. The payoffs of the players are transferable.


**Theorem B9. (Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)**

Consider the randomly furcating cooperative stochastic dynamic game Problem B.9 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(d_{\alpha}^{i})^{k}}(t, x_{\tau}^{*})$, for $i \in N$, $\tau \in [t_{k}, t_{k+1} \setminus t_{i} \in [t_{k}, t_{k+1}]$, $k \in \{0,1,2,\ldots,m-1\}$, and $\theta_{\alpha}^{i} \in \{\theta_1, \theta_2, \ldots, \theta_n\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_{i}^{(d_{\alpha}^{i})^{k}}(\tau) = - \left[ \xi^{(d_{\alpha}^{i})^{k}}(t, x_{\tau}^{*}) \right]_{\tau=\tau},$$

$$- \left[ \xi^{(d_{\alpha}^{i})^{k}}(t, x_{\tau}^{*}) \right]_{\tau=\tau} f[\tau, x_{\tau}^{*}, \psi_{1}^{(d_{\alpha}^{i})^{k}}(\tau, x_{\tau}^{*}), \psi_{2}^{(d_{\alpha}^{i})^{k}}(\tau, x_{\tau}^{*}), \ldots, \psi_{n}^{(d_{\alpha}^{i})^{k}}(\tau, x_{\tau}^{*})]$$

$$- \frac{1}{2} \sum_{k=1}^{m} \Omega_{\alpha}^{h_{\alpha}^{i}}(\tau, x_{\tau}^{*}) \left[ \xi^{(d_{\alpha}^{i})^{k}}(t, x_{\tau}^{*}) \right]_{\tau=\tau},$$

for $i \in N$ and $k \in \{1,2,\ldots,m\}$, given to player $i$ at time $\tau \in [t_{k}, t_{k+1}]$ contingent upon $\theta_{\alpha}^{i} \in \{\theta_1, \theta_2, \ldots, \theta_n\}$ has
occurred at time \( t_k \), leads to the realization of the imputation \( \xi^{(i,k)}(t,x^*) \), for \( i \in N \), \( \tau \in [t_k, t_{k+1}] \), \( t \in [\tau, t_{k+1}] \), \( k \in \{0,1,\ldots,m-1\} \), and \( \theta^k_{a_k} \in \{\theta_1, \theta_2, \ldots, \theta_n\} \).

where

\[
\{u_{i}^{(m)\theta^m_{a_k}}(t) = \psi_{i}^{(m)}(t,x), \text{ for } t \in [t_m, T] ; u_{i}^{(k)\theta^k_{a_k}}(t) = \psi_{i}^{(k)}(t,x), \text{ for } t \in [t_k, t_{k+1}] \},
\]

\( k \in \{0,1,\ldots,m-1\} \) and \( i \in N \), contingent upon the events \( \theta^m_{a_k} \) and \( \theta^k_{a_k} \) is a set of controls that provides a group optimal solution for the game Problem 11 yielding continuously differentiable functions \( W^{(\theta^m_{a_k})}_{(k)}(t,x) : [t_m, T] \times \mathbb{R}^k \to \mathbb{R} \) and \( W^{(\theta^k_{a_k})}_{(k)}(t,x) : [t_k, t_{k+1}] \times \mathbb{R}^k \to \mathbb{R} \) for \( k \in \{0,1,\ldots,m-1\} \) which satisfy the following partial differential equations:

\[
-W^{(\theta^m_{a_k})}_{(m)}(t,x) - \frac{1}{2} \sum_{i,h=1}^{n} \Omega^{h}_{i,m}(t,x) W^{(\theta^m_{a_k})}_{(m)}(t,x)
= \max_{u_{i}^{(m)\theta^m_{a_k}}, u_{i}^{(m)\theta^m_{a_k}}, \ldots, u_{i}^{(m)\theta^m_{a_k}}} \left\{ \sum_{j=1}^{n} g^{(m)\theta^m_{a_k}} \left[ t, x(t), u_{1}^{(m)\theta^m_{a_k}}, u_{2}^{(m)\theta^m_{a_k}}, \ldots, u_{n}^{(m)\theta^m_{a_k}} \right] e^{-\tau(t,t)} \right. \\
\left. + W^{(\theta^m_{a_k})}_{(m)}(t,x) \int \left[ t, x, u_{1}^{(m)\theta^m_{a_k}}, u_{2}^{(m)\theta^m_{a_k}}, \ldots, u_{n}^{(m)\theta^m_{a_k}} \right] \right\}, \text{ and}
\]

\[
-W^{(\theta^k_{a_k})}_{(k)}(t,x) - \frac{1}{2} \sum_{i,h=1}^{n} \Omega^{h}_{i,k}(t,x) W^{(\theta^k_{a_k})}_{(k)}(t,x)
= \max_{u_{i}^{(k)\theta^k_{a_k}}, u_{i}^{(k)\theta^k_{a_k}}, \ldots, u_{i}^{(k)\theta^k_{a_k}}} \left\{ \sum_{j=1}^{n} g^{(k)\theta^k_{a_k}} \left[ t, x(t), u_{1}^{(k)\theta^k_{a_k}}, u_{2}^{(k)\theta^k_{a_k}}, \ldots, u_{n}^{(k)\theta^k_{a_k}} \right] e^{-\tau(t-t)} \right. \\
\left. + W^{(\theta^k_{a_k})}_{(k)}(t,x) \int \left[ t, x, u_{1}^{(k)\theta^k_{a_k}}, u_{2}^{(k)\theta^k_{a_k}}, \ldots, u_{n}^{(k)\theta^k_{a_k}} \right] \right\}, \text{ and}
\]

\[
W^{(\theta^m_{a_k})}_{(m)}(t_k+1,x) = \sum_{a_k=1}^{n} \lambda_{a} \ W^{(\theta^m_{a_k})}_{(m)}(t_k+1,x), \text{ for } k \in \{0,1,\ldots,m-1\}.
\]

Problem B9.
Consider a class of randomly furcating cooperative stochastic differential game in which there are \( n \) players. The game interval is \([t_0, T]\). When the game commences at \( t_0 \), the payoff structures of the players in the interval \([t_0, t_1]\) are known. In future instants of time \( t_k \) \((k = 1, 2, \ldots, m)\), where \( t_0 < t_m < T = t_{m+1} \), the payoff structures in the time interval \([t_k, t_{k+1}]\) are affected by a series of random events \( \Theta^k \). In particular, \( \Theta^k \) for \( k \in \{1, 2, \ldots, m\} \) are independent and identically distributed random variables with range \( \{\theta_1, \theta_2, \ldots, \theta_n\} \) and corresponding probabilities \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). At time \( T \) a terminal value \( q'(x(T)) \) will be given to player \( i \). Player \( i \) seeks to maximize the expected payoff:

\[
E_{x_i} \left\{ \int_{t_0}^{t_1} g^{(i,k)}[s,x(s),u_1(s),u_2(s),\ldots,u_n(s)] e^{-\tau(s-t_0)} ds \right\}
\]
\[
\sum_{a_k=1}^{m_k} \sum_{k=1}^{n_k} \lambda_{a_k} \int_{t_k}^{t_{k+1}} g \left[ t, x(s), u_1(s), u_2(s), \ldots, u_n(s) \right] e^{-r(s-t_k)} + e^{-r(T-t_k)} q^i (x(T)) \, ds,
\]
for \( i \in \{1, 2, \ldots, n \} \equiv N \),

where \( x(s) \in X \subset \mathbb{R}^x \) is a vector of state variables, \( \theta^i_k \in \{ \theta_1, \theta_2, \ldots, \theta_k \} \) for \( k \in \{1, 2, \ldots, m_k\} \), \( \theta^i_0 = \theta^0 \) is known at time \( t_0 \), \( r \) is the discount rate, \( u_i \in U^i \) is the control of player \( i \), and \( E_{t_k} \) denotes the expectation operator performed at time \( t_0 \).

The payoffs of the players are transferable.

The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

\[
dx(s) = \left[ f(t, x(s), u_1(s), u_2(s), \ldots, u_n(s)) + \sigma(t, x(s)) \right] ds + \sigma(t, x(s)) \, dz(s),
\]

where \( \sigma(t, x(s)) \) is a \( \kappa \times \nu \) matrix and \( z(s) \) is a \( \nu \)-dimensional Wiener process and the initial state \( x_0 \) is given. Let \( \Omega[s, x(s)] = \sigma[s, x(s)] \sigma[s, x(s)]^T \) denote the covariance matrix with its element in row \( h \) and column \( \zeta \) denoted by \( \Omega_h^{\zeta} [s, x(s)] \).

\( u_i \in U_i \subset \text{comp}\mathbb{R}^i \) is the control vector of player \( i \), for \( i \in N \).

References:


Theorem B10. (Subgame-consistent PDP for Random-horizon Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the uncertain horizon randomly faceted cooperative stochastic dynamic game Problem B10 (see below) in which the players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation \( \xi^{(\sigma_k)} (k, x_k^* ) = [\xi^{(\sigma_1)} (k, x_1^* ), \xi^{(\sigma_2)} (k, x_2^* ), \ldots, \xi^{(\sigma_k)} (k, x_k^* )] \) along the cooperative trajectory given that \( \theta^i_k \) has occurred in stage \( k \), for \( \sigma_k \in \{1, 2, \ldots, \eta_k \} \) and \( k \in \{1, 2, \ldots, T\} \). A Payoff Distribution Procedure (PDP) with a payment equaling

\[
B^{(\sigma_k)} (x_k^* ) = \xi^{(\sigma_k)} (k, x_k^* ) - E_{\theta_k} \left\{ \frac{\partial \theta_k}{\partial \sigma_k} q_k \left[ f_k (x_k^*, \psi_k^{(\sigma_k)} (x_k^* )) + \partial \sigma_k \right] + \sum_{\sigma_k=1}^{\mu_k} \sum_{\sigma_k=1}^{\mu_k} \xi^{(\sigma_k)} (k + 1, f_k (x_k^*, \psi_k^{(\sigma_k)} (x_k^* )) + \partial \sigma_k ) \right\},
\]

given to player \( i \in N \) at stage \( k \in \{1, 2, \ldots, T\} \) if \( \theta^i_k \in \{ \theta^1_k, \theta^2_k, \ldots, \theta^\eta_k \} \) occurs would lead to the realization of the imputation:

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\[ \xi^{(\sigma_k)}(k,x_k^*) = [\xi^{(\sigma_k)}_1(k,x_k^*), \xi^{(\sigma_k)}_2(k,x_k^*), \ldots, \xi^{(\sigma_k)}_{\eta_k}(k,x_k^*)], \quad \text{for } \sigma_k \in \{1,2,\ldots,\eta_k\} \text{ and } k \in \{1,2,\ldots,T\}; \]

where
\[ \psi^{(\sigma_k)}_k(x_k^*) = [\psi^{(\sigma_k)}_k(x_k^*), \psi^{(\sigma_k)}_k(x_k^*), \ldots, \psi^{(\sigma_k)}_k(x_k^*)], \quad \text{for } k \in \mathcal{K} \text{ and } i \in \mathcal{N} \]

is a set of controls that provides a group optimal solution to the Problem B10 yielding functions \( W^{(\sigma)}(t,x) \), for \( \sigma \in \{1,2,\ldots,\eta\} \text{ and } t \in \{1,2,\ldots,T\} \), such that the following recursive relations are satisfied:
\[
W^{(\sigma_{t+1})}(T+1,x) = \sum_{j=1}^{n} q_{t+1}^j(x),
\]
\[
W^{(\sigma_{t})}(T,x) = \max_{u_1^t,u_2^t,\ldots,u_n^t} \mathbb{E}_{\sigma_{t}} \left\{ \sum_{j=1}^{n} g_j^t(x,u_1^t,u_2^t,\ldots,u_n^t;\Theta_{t}^{\sigma_{t}}) \right. \\
+ W^{(\sigma_{t+1})}(T+1,f_T(x,u_1^t,u_2^t,\ldots,u_n^t)+\Theta_T) \right\},
\]
\[
W^{(\sigma_{\tau})}(\tau,x) = \max_{u_1^\tau,u_2^\tau,\ldots,u_n^\tau} \mathbb{E}_{\sigma_{\tau}} \left\{ \sum_{j=1}^{n} g_j^\tau(x,u_1^\tau,u_2^\tau,\ldots,u_n^\tau;\Theta_{\tau}^{\sigma_{\tau}}) \right. \\
+ \sum_{j=1}^{n} \sigma_{\tau}^j q_{\tau}^j[f_T(x,u_1^\tau,u_2^\tau,\ldots,u_n^\tau)+\Theta_T] \right. \\
+ \sum_{\tau+1}^{\tau} \sigma_{\tau+1}^j \sum_{\tau+1}^{\eta_k} \chi_{\tau+1}^{(\sigma_{\tau+1})} W^{(\sigma_{\tau+1})}[(\tau+1,f_T(x,u_1^\tau,u_2^\tau,\ldots,u_n^\tau)+\Theta_T) \right\}
\]
for \( \tau \in \{1,2,\ldots,T-1\} \).

**Problem B10.**

Consider the \( \hat{T} \)-stage uncertain horizon randomly furcating cooperative stochastic dynamic game problem where \( \hat{T} \) is a random variable with range \{1,2,\ldots,T\} and corresponding probabilities \( \{\sigma_1,\sigma_2,\ldots,\sigma_{\hat{T}}\} \). Conditional upon the reaching of stage \( \tau \), the probability of the game would last up to stages \( \tau, \tau+1, \ldots, T \) becomes respectively
\[
\frac{\sigma_1}{\sum_{\tau=1}^{\tau} \sigma_1}, \frac{\sigma_2}{\sum_{\tau=1}^{\tau} \sigma_2}, \ldots, \frac{\sigma_{\hat{T}}}{\sum_{\tau=1}^{\tau} \sigma_{\hat{T}}}. \tag{1.1}
\]

The state space of the game is \( X \in \mathbb{R}^m \) and the state dynamics of the game is characterized by the stochastic difference equation:
\[
x_{k+1} = f_k(x_k,u_1^k,u_2^k,\ldots,u_n^k)+\Theta_k, \tag{1.2}
\]
for \( k \in \{1,2,\ldots,T\} \) and \( x_1 = x_0 \),
where \( u_i^k \in U^i \subset \mathbb{R}^m \) is the control vector of player \( i \) at stage \( k \), \( x_k \in X \) is the state, and \( \Theta_k \) is a sequence of statistically independent random variables.
The payoff of player $i$ at stage $k$ is $g_k^i[x_k^1,u_k^1,u_k^2,\ldots,u_k^n;\theta_k]$ which is affected by a random variable $\theta_k$. In particular, $\theta_k$ for $k \in \{1,2,\ldots,T\}$ are independent random variables with range $\{\theta_1^k,\theta_2^k,\ldots,\theta_T^k\}$ and corresponding probabilities $\{\lambda_1^k,\lambda_2^k,\ldots,\lambda_T^k\}$. In stage 1, it is known that $\theta_1^i$ equals $\theta_1^i$ with probability $\lambda_1^1 = 1$. When the game ends after stage $\hat{T}$, a terminal payment $q_{\hat{T}+1}^i(x_{\hat{T}+1})$ will be given to player $i$ in stage $\hat{T} + 1$.

The objective that player $i$ seeks to maximize is

$$E_{\theta_1,\theta_2,\ldots,\theta_T}(\sum_{k=1}^{T} \sigma_k \left[ \sum_{k'=1}^{T} g_k^i[x_k^1,u_k^1,u_k^2,\ldots,u_k^n;\theta_k] + q^i(x_{\hat{T}+1}) \right]),$$

for $i \in \{1,2,\ldots,n\} = N$, where $E_{\theta_1,\theta_2,\ldots,\theta_T}$ is the expectation operation with respect to the random variables $\theta_1,\theta_2,\ldots,\theta_T$. Since there is no uncertainty in the payoff structure in stage $T+1$, we denote $\sigma_{T+1} = 1$, $\theta_{T+1} = \theta_{T+1}^1$ with probability $\lambda_{T+1}^1 = \lambda_{T+1}^1 = 1$ for notational convenience.


**Theorem B11.** (Subgame-consistent Solution Mechanism for Dynamic Cooperation under Non-transferrable Payoffs (NTU))

Consider the non-transferrable payoff/utility (NTU) cooperative dynamic game Problem B11 (see below) in which the players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1,\hat{\alpha}_k^2,\ldots,\hat{\alpha}_k^n), \text{ for } k \in \kappa\}$ for joint maximization of the weighted joint payoff so that the imputation $\xi^i(k,x_k^i)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^i\}_{k=1}^T$ can be achieved.

A set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1,\hat{\alpha}_k^2,\ldots,\hat{\alpha}_k^n), \text{ for } k \in \kappa\}$ and a set of strategies $\{\psi_k^{(\hat{\alpha}_k)}(x), \text{ for } k \in \kappa \text{ and } i \in N\}$ provides a subgame consistent solution to the NTU cooperative dynamic game Problem B11 if there exist functions $W^{(\hat{\alpha}_k)}(k,x)$ and $W^{(\hat{\alpha}_k)}(k,x)$, for $i \in N, k \in \kappa$, which satisfy the following recursive relations:

$$W^{(\hat{\alpha}_k)}(T+1,x) = q^i(x_{\hat{T}+1}),$$

$$W^{(\hat{\alpha}_k)}(k,x) = \max_{u_k^1,u_k^2,\ldots,u_k^n} \left\{ \sum_{j=1}^{n} \hat{\alpha}_k^j g_k^j(x,u_k^1,u_k^2,\ldots,u_k^n) + \sum_{j=1}^{n} \hat{\alpha}_k^j W^{(\hat{\alpha}_k)}[k+1,f_k(x,u_k^1,u_k^2,\ldots,u_k^n)] \right\};$$

$$W^{(\hat{\alpha}_k)}(k,x) = g_k^1(x,\psi_k^{(\hat{\alpha}_k)^1}(x),\psi_k^{(\hat{\alpha}_k)^2}(x),\ldots,\psi_k^{(\hat{\alpha}_k)^n}(x)) + W^{(\hat{\alpha}_k)}(k+1,f_k(x,\psi_k^{(\hat{\alpha}_k)^1}(x),\psi_k^{(\hat{\alpha}_k)^2}(x),\ldots,\psi_k^{(\hat{\alpha}_k)^n}(x))],$$

for $i \in N$ and $k \in \kappa$. 

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and imputation \( \xi^i(k,x^*_k) \) for player \( i \in N \) in stage \( k \in \kappa \).

The value function \( W^{(\hat{a}_i)}(k,x) \) is the payoff for player \( i \in N \) in stage \( k \in \kappa \) under cooperation.

**Problem B11.**
Consider the general \( T \)–stage \( n \)–person dynamic game with initial state \( x_1^0 \). The state space of the game is \( X \in \mathbb{R}^n \) and the state dynamics of the game is characterized by the difference equation:
\[
x_{k+1} = f_k(x_k, u^1_k, u^2_k, \ldots, u^n_k),
\]
for \( k \in \{1,2,\cdots,T\} \equiv \kappa \) and \( x_1 = x_1^0 \), where \( u^i_k \in \mathbb{R}^m \) is the control vector of player \( i \) at stage \( k \), and \( x_k \in X \) is the state of the game. The payoff that player \( i \) seeks to maximize is
\[
\sum_{k=1}^T g^i_k(x_k, u^1_k, u^2_k, \ldots, u^n_k) + q^i(x_{T+1}),
\]
for \( i \in \{1,2,\cdots,n\} \equiv N \),
where \( q^i(x_{T+1}) \) is the terminal payoff that player \( i \) will receive in stage \( T+1 \).

The payoffs of the players are not transferable. The players agree to use a set of payoff weights \( \{\hat{\alpha}_k = (\hat{\alpha}^1_k, \hat{\alpha}^2_k, \ldots, \hat{\alpha}^n_k)\} \), for \( k \in \kappa \) for joint maximization of the expected weighted joint payoff so that the imputation \( \xi^i(k,x^*_k) \) for player \( i \in N \) in stage \( k \in \kappa \) along the cooperative trajectory \( \{x^*_k\}_{k=1}^T \) can be achieved.


**Theorem B12.** (Subgame-consistent Solution Mechanism for Stochastic Dynamic Cooperation under Non-transferrable Payoffs (NTU))
Consider the non-transferrable payoff/utility (NTU) cooperative stochastic dynamic game Problem B12 (see below) in which the players agree to use a set of payoff weights \( \{\hat{\alpha}_k = (\hat{\alpha}^1_k, \hat{\alpha}^2_k, \ldots, \hat{\alpha}^n_k)\} \), for \( k \in \kappa \) for joint maximization of the expected weighted joint payoff so that the imputation \( \xi^i(k,x^*_k) \) for player \( i \in N \) in stage \( k \in \kappa \) along the cooperative trajectory \( \{x^*_k\}_{k=1}^T \) can be achieved.

A set of payoff weights \( \{\psi^i_k(\hat{\alpha}_k, \hat{\alpha}_k^2, \ldots, \hat{\alpha}_k^n)\} \), for \( k \in \kappa \) and \( i \in N \) provides a subgame consistent solution to the NTU cooperative dynamic game Problem B12 if there exist functions \( W^{(\hat{a}_i)}(k,x) \) and \( W^{(\hat{a}_i)}(k,x) \), for \( i \in N \ k \in \kappa \), which satisfy the following recursive relations:
\[
W^{(\hat{a}_r,i)}(T+1,x) = q^i(x_{T+1}),
\]
\[
W^{(\hat{\alpha}_k)}(k,x) = \max_{u_k^1, \ldots, u_k^n} \left\{ E_{\hat{\alpha}_k} \left[ \sum_{j=1}^{n} \hat{\alpha}_j g_k^j(x, u_k^1, u_k^2, \ldots, u_k^n) + \sum_{j=1}^{n} \hat{\alpha}_j W^{(\hat{\alpha}_k)}[k+1, f_k(x, u_k^1, u_k^2, \ldots, u_k^n) + G_k(x) \theta_k] \right] \right\};
\]

\[
W^{(\hat{\alpha}_i)}(k,x) = E_{\hat{\alpha}_i} \left\{ g_k^i(x, \psi_k^{(\hat{\alpha}_i)}(x), \psi_k^{(\hat{\alpha}_i)^2}(x), \ldots, \psi_k^{(\hat{\alpha}_i)^n})(x) + W^{(\hat{\alpha}_i)}[k+1, f_k(x, \psi_k^{(\hat{\alpha}_i)}(x), \psi_k^{(\hat{\alpha}_i)^2}(x), \ldots, \psi_k^{(\hat{\alpha}_i)^n})(x) + G_k(x) \theta_k] \right\},
\]

for \( i \in N \) and \( k \in \kappa \); and imputation \( \xi^i(k, x_k^*) \) for player \( i \in N \) in stage \( k \in \kappa \).

The value function \( W^{(\hat{\alpha}_i)}(k,x) \) is the expected payoff for player \( i \in N \) in stage \( k \in \kappa \) under cooperation. \( \blacksquare \)

**Problem B12.**
Consider the NTU cooperative stochastic dynamic game with initial state \( x_1^0 \). The state space of the game is \( X \in \mathbb{R}^n \) and the state dynamics of the game is characterized by the stochastic difference equation:

\[
x_{k+1} = f_k(x_k^1, u_k^1, u_k^2, \ldots, u_k^n) + G_k(x_k) \theta_k,
\]

for \( k \in \{1, 2, \ldots, T\} \equiv \kappa \) and \( x_1 = x_1^0 \),

where \( u_k^i \in \mathbb{R}^m \) is the control vector of player \( i \) at stage \( k \), and \( x_k \in X \) is the state of the game and \( \theta_k \) is a set of independent random variable. The payoff that player \( i \) seeks to maximize is

\[
E_{\hat{\alpha}_i, \theta_2, \ldots, \theta_T} \left\{ \sum_{k=1}^{T} g_k^i(x_k^1, u_k^1, u_k^2, \ldots, u_k^n, x_k, x_{k+1}) + q'(x_{T+1}) \right\},
\]

for \( i \in \{1, 2, \ldots, n\} \equiv N \),

where \( q'(x_{T+1}) \) is the terminal payoff that player \( i \) will received in stage \( T + 1 \), and \( E_{\hat{\alpha}_i, \theta_2, \ldots, \theta_T} \) is the expectation operation with respect to the statistics of \( \theta_1, \theta_2, \ldots, \theta_T \).

The payoffs of the players are not transferable. The payoffs of the players are not transferable. The players agree to use a set of payoff weights \( \{ \hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \ldots, \hat{\alpha}_k^n), \text{for} k \in \kappa \} \) for joint maximization of the expected weighted joint payoff so that the imputation \( \xi^i(k, x_k^*) \) for player \( i \in N \) in stage \( k \in \kappa \) along the cooperative trajectory \( \{ x_k^* \}_{k=1}^{T} \) can be achieved.


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**Part C: Identities and Equations in Economics**

**C1. Inter-temporal Roy’s Identity**

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\[ \frac{\partial v}{\partial p} (W_i^0, p, p_{i+1}, \ldots, p_T) + \frac{\partial v}{\partial W} (W_i^0, p, p_{i+1}, \ldots, p_T) \]
\[ = -(1 + r)^{-(h-i)} \phi_h (W_h^0, p_h, p_{h+1}, \ldots, p_T); \]
or in an alternative form
\[ \frac{\partial v}{\partial p} (W_i^0, p, p_{i+1}, \ldots, p_T) + \delta_h \frac{\partial v}{\partial W} (W_h^0, p_h, p_{h+1}, \ldots, p_T) \]
\[ = -\phi_h (W_h^0, p_h, p_{h+1}, \ldots, p_T); \]
for \( \ell \in \{1, 2, \ldots, T\}, \ h \in \{\ell, \ell + 1, \ldots, T\} \) and \( j \in \{1, 2, \ldots, n_h\}, \)

where
\[ W_i = W_i^0, \]
\[ W_{i+1} = (1 + r)(W_i^0 - p, \phi_i) + Y_{i+1}, \]
\[ W_{i+2} = (1 + r)(W_{i+1}^0 - p_{i+1} \phi_{i+1}) + Y_{i+2}, \]
\[ \vdots \]
\[ W_T = (1 + r)(W_{T-1}^0 - p_{T-1} \phi_{T-1}) + Y_T. \]

**Problem C.1.**

The inter-temporal Roy’s identity is derived from the consumer problem in which the consumer maximizes his inter-temporal utility

\[ u^i(x^1, x^2, \ldots, x^n) + \sum_{k=2}^{T} \delta_k u^k(x^1, x^2, \ldots, x^n) \]
\[ = u^i(x) + \sum_{k=2}^{T} \delta_k u^k(x) = \sum_{k=1}^{T} \delta_k u^k(x) \]

subject to the budget constraint characterized by the wealth dynamics

\[ W_{k+1} = W_k - \sum_{h=1}^{n_h} p^h x^h_k + r(W_k - \sum_{h=1}^{n_h} p^h x^h_k) + Y_{k+1}, \quad W_1 = W_1^0, \]

where
\[ x_k = (x^1_k, x^2_k, \ldots, x^n_k) \]
is the vector of quantities of goods consumed in period \( k \),
\[ p_k = (p^1_k, p^2_k, \ldots, p^n_k) \]
is price vector, \( r \) is the interest rate, \( Y_k \) is the income that the consumer will receive in period \( k \), \( \delta_k = \left( \prod_{c=2}^{k} \beta_c \right) \) is the discount factor with \( \beta_c \) being the consumer’s subjective one-period discount factor for the duration from period \( \tau - 1 \) to period \( \tau \), \( \beta_1 = 1 \) for the discount factor in the initial period 1 and \( \delta_k = \left( \prod_{c=1}^{k} \beta_c \right) = \left( \prod_{c=2}^{k} \beta_c \right) \). The period \( k \) utility function \( u^k(x^1_k, x^2_k, \ldots, x^n_k) \) is continuously differentiable and quasi-concave yielding convex level (indifference) curves. The time preference factor is embodied in the utility function. The time preference factor is embodied in the utility function. The amount of unconsumed wealth \( W_k - p_k x_k \) in period \( k \) will generate an interest income \( r(W_k - p_k x_k) \) in period \( k + 1 \).
In addition, \( v^f(W^0_t, p_t, p_{t+1}, \ldots, p_T) \) is the intertemporal indirect utility at period \( \ell \), and \( \phi_h(W^0_h, p_h, p_{h+1}, \ldots, p_T) \) is the ordinary demand function of commodity \( j \) in period \( h \).

**References:**


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**C2. Inter-temporal Roy’s Identity under Stochastic Income**

\[
\frac{\partial v^f(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial p^j_t} + \frac{\partial v^f(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial W^0_t} \\
= -\frac{\partial \phi^j(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial W^0_t}, \text{ for } j \in \{1, 2, \ldots, n_t\};
\]

\[
\frac{\partial v^f(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial W^0_t} = -\sum_{h=1}^{m_t} \lambda^h_t \sum_{j=1}^{m_t} \lambda^{j,h} \sum_{h=1}^{m_t} \lambda^{j,h} \delta^{h}_{t+1} \frac{\partial v^h(W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}, p)}{\partial W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}} \\
\times \frac{\partial v^h(W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}, p)(1+r)^{-(h-t)}}{\partial W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}},
\]

for \( \ell \in \{1, 2, \ldots, T\} \), \( h \in \{\ell + 1, \ell + 2, \ldots, T\} \), and \( j \in \{1, 2, \ldots, n_t\} \), and \( v^h(W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}, p) \) is the short form for \( v^h(W^0_{t+1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{t+1}, p_h, p_{h+1}, \ldots, p_T) \), where

\[
W^0_t = W^0_\ell,
\]

\[
W^0_{t+1} \theta^{j,h}_{t+1} = (1+r)[W^0_t - p_t \phi_h(W^0_t, p)] + \theta^{j,h}_{t+1},
\]

\[
W^0_{t+2} \theta^{j,h}_{t+2} = (1+r)[W^0_{t+1} - p_{t+1} \phi_h(W^0_{t+1}, p)] + \theta^{j,h}_{t+2},
\]

\[
\vdots
\]

\[
W^0_T \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{T-1} = (1+r)[W^0_{T-1} \theta^{j,h}_{t+1} \cdots \theta^{j,h}_{T-1} - p_{T-1} \phi_h(W^0_{T-1}, p_{T-1})] + \theta^{j,h}_{T}.
\]

---

**Problem C2.**

The inter-temporal Roy’s identity under stochastic income is derived from the consumer problem in which the consumer maximizes his expected inter-temporal utility

\[
E_{\theta_h, \phi_h \cdots \phi_T} \left\{ \sum_{k=1}^{T} \delta^k u^k(x^1_k, x^2_k, \ldots, x^n_k) \right\} = E_{\theta_h, \phi_h \cdots \phi_T} \left\{ \sum_{k=1}^{T} \delta^k u^k(x^k_k) \right\}
\]

subject to the budget constraint characterized by the stochastic wealth dynamics

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\[ W_{k+1} = (1 + r)(W_k - p_k x_k) + \theta_{k+1}, \quad W_1 = W_0^\ast, \]

where

\[ \theta_k \] is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \ldots, T\} \), is a set of statistically independent random variables, and \( E_{\theta_0, \theta_1, \ldots, \theta_T} \) is the expectation operation with respect to the statistics of \( \theta_0, \theta_1, \ldots, \theta_T \). The random variable \( \theta_k \) has a non-negative range \( \{\theta_1^0, \theta_2^0, \ldots, \theta_T^0\} \) with corresponding probabilities \( \{\lambda_1^0, \lambda_2^0, \ldots, \lambda_T^0\} \), for \( k \in \{2, \ldots, T\} \).

**References:**


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**C.3. Inter-temporal Roy’s Identity under Stochastic Life-span**

\[
\frac{\partial V^\ast(W_0^0, p_1, p_{i+1}, \ldots, p_T)}{\partial p_h^0} + \frac{\partial V^\ast(W_1^0, p_1, p_{i+1}, \ldots, p_T)}{\partial W_1^0} = -(1 + r)^{-(h-i)} \varphi_h^0(W_0^0, p_h, p_{i+1}, \ldots, p_T);
\]

or in an alternatively form:

\[
\frac{\partial V^\ast(W_0^0, p_1, p_{i+1}, \ldots, p_T)}{\partial p_h^0} + \delta_{i+1}^h \frac{\partial V^\ast(W_0^0, p_h, p_{i+1}, \ldots, p_T)}{\partial W_0^0} = \sum_{\zeta=1}^{T} \gamma_{\zeta} \varphi_h^0(W_0^0, p_h, p_{i+1}, \ldots, p_T);
\]

for \( \ell \in \{1, 2, \ldots, T\} \) \( h \in \{\ell, \ell + 1, \ldots, T\} \) and \( j \in \{1, 2, \ldots, n_h\} \), where

\[
W_\ell = W_0^0, \\
W_{\ell+1} = (1 + r)(W_\ell^0 - p_\ell \varphi_j) + Y_{\ell+1}, \\
W_{\ell+2} = (1 + r)(W_{\ell+1}^0 - p_{\ell+1} \varphi_j) + Y_{\ell+2}, \\
\vdots \\
W_h^0 = (1 + r)(W_{h-1}^0 - p_{h-1} \varphi_j) + Y_h.
\]

**Problem C3.**

The inter-temporal Roy’s identity under stochastic life-span is derived from the consumer problem in which the consumer’s life-span involves \( \hat{T} \) periods where \( \hat{T} \) is a random variable with range \( \{1, 2, \ldots, T\} \) and corresponding probabilities \( \{\gamma_1, \gamma_2, \ldots, \gamma_T\} \). Conditional upon the reaching of period \( \tau \), the probability of the consumer’s life-span would last up to periods \( \tau, \tau + 1, \ldots, T \) becomes respectively
\[
\begin{align*}
\frac{\gamma_{r}}{T} , \frac{\gamma_{r+1}}{T} , \ldots , \frac{\gamma_{T}}{T}. \\
\sum_{\zeta=\ell}^{\gamma_{\zeta}} \sum_{\zeta=\ell}^{\gamma_{\zeta}} \sum_{\zeta=\ell}^{\gamma_{\zeta}}.
\end{align*}
\]

The consumer maximizes his expected inter-temporal utility

\[
\sum_{\tau=1}^{T} \gamma_{\tau} \sum_{k=1}^{h} \delta_{k} u^{k}(x_{k}),
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{k+1} = W_{k} - \sum_{h=1}^{n_{h}} p^{h}_{k} x^{h}_{k} + r(W_{k} - \sum_{h=1}^{n_{h}} p^{h}_{k} x^{h}_{k}) + Y_{k+1}, \quad W_{1} = W_{0}^{0}.
\]

where

\[r\] is the interest rate, \(Y_{k}\) is the income that the consumer will receive in period \(k\).


C4. Inter-temporal Roy’s Identity under Stochastic Income and Life-span

\[
\begin{align*}
\frac{\partial v^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T})}{\partial p_{j}^{h}} + \frac{\partial v^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T})}{\partial W_{t}^{0}}
\end{align*}
\]

\[
\equiv -\phi_{t}^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T}), \text{ for } j \in \{1,2,\ldots,n_{j}\},
\]

\[
\begin{align*}
\frac{\partial v^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T})}{\partial W_{t}^{0}} = \frac{\partial v^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T})}{\partial W_{t}^{0}}
\end{align*}
\]

\[
\equiv - \sum_{j=1}^{n_{j}} \lambda_{j_{1}}^{j} \sum_{j_{2}=1}^{n_{j}} \lambda_{j_{2}}^{j} \ldots \sum_{j_{h}=1}^{n_{j}} \lambda_{j_{h}}^{j} \delta^{j_{k}} h_{k} \delta_{t_{k+1}}^{j_{k}} (W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p) \cdot \frac{\partial v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p)}{\partial W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}}, (1 + r)^{-h-t}
\]

\[
\begin{align*}
\times \left[ \sum_{j_{1}=1}^{n_{j_{1}}} \sum_{j_{2}=1}^{n_{j_{2}}} \lambda_{j_{1}}^{j_{2}} \sum_{j_{3}=1}^{n_{j_{3}}} \lambda_{j_{2}}^{j_{3}} \ldots \sum_{j_{h}=1}^{n_{j_{h}}} \lambda_{j_{h}}^{j_{h}} \delta^{j_{k}} h_{k} \delta_{t_{k+1}}^{j_{k}} (W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p) \cdot \frac{\partial v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p)}{\partial W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}}, (1 + r)^{-h-t}\right];
\end{align*}
\]

or in an alternative form:

\[
\begin{align*}
\frac{\partial v^{j}(W_{t}^{0}, p_{t}, p_{t+1}, \ldots, p_{T})}{\partial p_{h}^{j}}
\end{align*}
\]

\[
\equiv - \sum_{j_{1}=1}^{n_{j_{1}}} \sum_{j_{2}=1}^{n_{j_{2}}} \lambda_{j_{1}}^{j_{2}} \sum_{j_{3}=1}^{n_{j_{3}}} \lambda_{j_{2}}^{j_{3}} \ldots \sum_{j_{h}=1}^{n_{j_{h}}} \lambda_{j_{h}}^{j_{h}} \delta^{j_{k}} h_{k} \delta_{t_{k+1}}^{j_{k}} (W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p) \cdot \frac{\partial v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p)}{\partial W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}}, (1 + r)^{-h-t}
\]

\[
\begin{align*}
\sum_{j=1}^{n_{j}} \lambda_{j_{1}}^{j} \sum_{j_{2}=1}^{n_{j}} \lambda_{j_{2}}^{j} \ldots \sum_{j_{h}=1}^{n_{j}} \lambda_{j_{h}}^{j} \delta^{j_{k}} h_{k} \delta_{t_{k+1}}^{j_{k}} (W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p) \cdot \frac{\partial v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p)}{\partial W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}};
\end{align*}
\]

for \(\ell \in \{1,2,\ldots,T\}, h \in \{\ell + 1, \ell + 2, \ldots, T\}\) and \(k \in \{1,2,\ldots,n_{h}\}\),

and \(v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p)\) is the short form for \(v^{j}(W_{h}^{\theta^{j_{1}}_{k_{1}} \theta^{j_{2}}_{k_{2}} \ldots \theta^{j_{h}}_{k_{h}}}, p_{h}, p_{h+1}, \ldots, p_{T})\),

where
\[ W_t = W_t^0, \]
\[ W_{t+1}^{t_0,t_1,t_2} = (1+r)[W_t^0 - p_t \phi_t(W_t^0, p)] + \theta_{t+1}, \]
\[ W_{t+2}^{t_0,t_1,t_2} = (1+r)[W_{t+1}^{t_0,t_1,t_2} - p_{t+1} \phi_{t+1}(W_{t+1}^{t_0,t_1,t_2}, p)] + \theta_{t+2}, \]
\[ \vdots \]
\[ W_T^{t_0,t_1,t_2,\ldots,t_n} = (1+r)[W_{T-1}^{t_0,t_1,t_2,\ldots,t_n} - p_{T-1} \phi_{T-1}(W_{T-1}^{t_0,t_1,t_2,\ldots,t_n}, p)] + \theta_T. \]

**Problem C4.**

The inter-temporal Roy’s identity under stochastic income and life-span is derived from the consumer problem in which the consumer’s life-span involves \( \hat{T} \) periods where \( \hat{T} \) is a random variable with range \( \{1, 2, \ldots, T\} \) and corresponding probabilities \( \{\gamma_1, \gamma_2, \ldots, \gamma_T\} \). Conditional upon the reaching of period \( \tau \), the probability of the consumer’s life-span would last up to periods \( \tau, \tau + 1, \ldots, T \) becomes respectively:

\[
\sum_{\tau = \tau}^{T} \gamma_{\tau}, \quad \sum_{\tau = \tau + 1}^{T} \gamma_{\tau + 1}, \quad \ldots, \quad \sum_{\tau = T}^{T} \gamma_{T}.
\]

The consumer maximizes his expected inter-temporal utility

\[
E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{\tau = 1}^{\hat{T}} \sum_{k=1}^{T} \delta^k u^k(x_k) \right\},
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{t+1} = W_t - \sum_{h=1}^{n_t} p_t^h x_t^h + r(W_t - \sum_{h=1}^{n_t} p_t^h x_t^h) + \theta_{t+1}, \quad W_1 = W_1^0.
\]

where \( \theta_k \) is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \ldots, T\} \), is a set of statistically independent random variables, and \( E_{\theta_1, \theta_2, \ldots, \theta_T} \) is the expectation operation with respect to the statistics of \( \theta_2, \theta_1, \ldots, \theta_T \). The random variable \( \theta_k \) has a non-negative range \( \{\theta_1^k, \theta_2^k, \ldots, \theta_m^k\} \) with corresponding probabilities \( \{\xi_1^k, \xi_2^k, \ldots, \xi_m^k\} \), for \( k \in \{2, \ldots, T\} \).

**References:**


**C5. Inter-temporal Roy’s Identity under Stochastic Preferences**

\[
\frac{\partial v^{(t_0)}(W_t^0, p)}{\partial p_t^j} + \frac{\partial v^{(t_0)}(W_t^0, p)}{\partial W_t^0} \equiv -\phi_t^{(t_0)}(W_t^0, p), \quad \text{for } j \in \{1, 2, \ldots, n_t, \}
\]

\[
\frac{\partial v^{(t_0)}(W_t^0, p)}{\partial p_t^h} + \frac{\partial v^{(t_0)}(W_t^0, p)}{\partial W_t^0} \equiv -\sum_{t_1=1}^{m_t} \rho_t^{(t_1)} \sum_{t_2=1}^{m_t} \rho_t^{(t_2)} \ldots
\]
... \sum_{\sigma_{t+1}}^{m} \rho_{h}^{\nu_{t}} \delta_{t+1}^{h} \left( W_{h}^{(s_{t+1}^{-1},s_{t+2}^{-1}, \ldots, s_{t+1}^{m})} + \frac{\partial h^{(s_{t+1})}}{\partial W_{h}^{(s_{t+1}^{-1},s_{t+2}^{-1}, \ldots, s_{t+1}^{m})}} \right) \phi_{h}^{(s_{t+1})} \left( W_{h}^{(s_{t+1}^{-1},s_{t+2}^{-1}, \ldots, s_{t+1}^{m})} , p \right) (1 + r)^{-(n-\ell)} \\
= \left[ \sum_{\sigma_{t+1}}^{m} \rho_{h}^{\nu_{t}} \delta_{t+1}^{h} \sum_{\sigma_{t+2}}^{m} \rho_{h}^{\nu_{t}} \delta_{t+2}^{h} \sum_{\sigma_{t+3}}^{m} \rho_{h}^{\nu_{t}} \delta_{t+3}^{h} \right] \frac{\partial v^{h(s_{t+1})}}{\partial W_{h}^{(s_{t+1}^{-1},s_{t+2}^{-1}, \ldots, s_{t+1}^{m})}} (1 + r)^{n-\ell} \right] ;

or in an alternative form

\frac{\partial v^{h(s_{t+1})}}{\partial W_{t}} \equiv - \sum_{\sigma_{t+1}}^{m} \rho_{h}^{\nu_{t}} \sum_{\sigma_{t+2}}^{m} \rho_{h}^{\nu_{t}} \sum_{\sigma_{t+3}}^{m} \rho_{h}^{\nu_{t}} \\
... \sum_{\sigma_{t+1}}^{m} \rho_{h}^{\nu_{t}} \delta_{t+1}^{h} \frac{\partial v^{h(s_{t+1})}}{\partial W_{h}^{(s_{t+1}^{-1},s_{t+2}^{-1}, \ldots, s_{t+1}^{m})}} (1 + r)^{n-\ell} ;

for \( \ell \in \{1,2,\ldots,T\} \), \( h \in \{\ell + 1, \ell + 2, \ldots,T\} \), \( k \in \{1,2,\ldots,n_{k}\} \) and \( \nu_{t} \in \{1,2,\ldots,m_{t}\} \),

where \( W_{t} = W_{t}^{0} \),

\( W_{t+1}^{U} = (1 + r)W_{t}^{0} - p_{t} \phi^{(U)}(W_{t}^{0}, p) + Y_{t+1} \),

\( W_{t+2}^{U_{t+1}} = (1 + r)W_{t+1}^{U} - p_{t+1} \phi^{(U_{t+1})}(W_{t+1}^{U}, p) + Y_{t+2} \),

if preference is \( u^{(U_{t+1})}(x_{t+1}) \) in period \( \ell + 1 \);

\( W_{t+3}^{U_{t+2}} = (1 + r)W_{t+2}^{U_{t+1}} - p_{t+2} \phi^{(U_{t+2})}(W_{t+2}^{U_{t+1}} , p) + Y_{t+3} \),

if preference is \( u^{(U_{t+2})}(x_{t+2}) \) in period \( \ell + 2 \);

\vdots

\( W_{t+T-1}^{U_{t+1}} = (1 + r)W_{t+T-2}^{U_{t+1}} - p_{T-1} \phi^{(U_{t+1})}(W_{t+T-2}^{U_{t+1}} , p) + Y_{t+T-1} \),

if preference is \( u^{(U_{t+1})}(x_{t+1}) \) in period \( T - 1 \);

\( W_{t+T}^{U_{t+1}} = (1 + r)W_{t+T-1}^{U_{t+1}} - p_{T} \phi^{(U_{t+1})}(W_{t+T-1}^{U_{t+1}} , p) + Y_{t+T} \);

Problem C5.
The inter-temporal Roy’s identity under stochastic preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be \( u^{(1)}(x_{1}) \). His future preferences are not known with certainty. In particular, his utility function in period \( k \in \{2,3,\ldots,T\} \) is known to be \( u^{(U_{k})}(x_{k}) \) with probability \( \rho_{k}^{\nu_{k}} \) for \( \nu_{k} \in \{1,2,\ldots,m_{k}\} \). We use \( \tilde{U}_{k} \) to denote the random variable with range \( \nu_{k} \in \{1,2,\ldots,m_{k}\} \) and corresponding probabilities \( \{\rho_{k}^{1}, \rho_{k}^{2}, \ldots, \rho_{k}^{m_{k}}\} \). The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

\[ E_{\theta_{1}, \theta_{2}, \ldots, \theta_{T}} \left\{ \sum_{k=1}^{T} \sum_{\nu_{k}=1}^{m_{k}} \rho_{k}^{\nu_{k}} \delta_{k}^{U_{k}} u^{(U_{k})}(x_{k}) \right\} \]

\[ = E_{\theta_{1}, \theta_{2}, \ldots, \theta_{T}} \left\{ u^{(1)}(x_{1}) + \sum_{k=2}^{T} \sum_{\nu_{k}=1}^{m_{k}} \rho_{k}^{\nu_{k}} \delta_{k}^{U_{k}} u^{(U_{k})}(x_{k}) \right\} \]

subject to the budget constraint characterized by the wealth dynamic

\[ W_{k+1} = W_{k} - \sum_{h=1}^{n_{k}} p_{h}^{h} x_{k}^{h} + r(W_{k} - \sum_{h=1}^{n_{k}} p_{h}^{h} x_{k}^{h}) + Y_{k+1}, \quad W_{1} = W_{1}^{0} \].

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C6. Inter-temporal Roy’s Identity under Stochastic Life-span and Preferences

\[
\frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial p_t^j} + \frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial W_t^0} \equiv -\phi_t^{(j)}(W_t^0, p), \text{ for } j \in \{1,2,\cdots, n_t\},
\]

\[
\frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial p_h^k} + \frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial W_t^0} \equiv -m_1^{-1} \sum_{t_1=1}^{m_1} \rho_t^{h_1} \sum_{t_2=1}^{m_2} \rho^{h_2} \cdots \]

\[
\left( \sum_{m_1=1}^{m_1} \rho^{h_1} \sum_{m_2=1}^{m_2} \rho^{h_2} \cdots \sum_{m_n=1}^{m_n} \rho^{h_n} \right) \frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial W_t^{\sigma^{(j)}_{\ell+1}}},
\]

or in an alternative form

\[
\frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial W_t^0} \equiv -\frac{\sum_{t=1}^{T} \gamma_t}{\sum_{t=1}^{T} \gamma_t} \sum_{t_1=1}^{m_1} \rho_t^{h_1} \sum_{t_2=1}^{m_2} \rho^{h_2} \cdots
\]

\[
\left( \sum_{m_1=1}^{m_1} \rho^{h_1} \sum_{m_2=1}^{m_2} \rho^{h_2} \cdots \sum_{m_n=1}^{m_n} \rho^{h_n} \right) \frac{\partial v^{(t_{j})}(W_t^0, p)}{\partial W_t^{\sigma^{(j)}_{\ell+1}}},
\]

for \( \ell \in \{1,2,\cdots,T\}, h \in \{\ell+1, \ell+2, \cdots, T\}, k \in \{1,2,\cdots, n_h\} \) and \( \nu_j \in \{1,2,\cdots, m_j\} \),

where

\[
W_t^0 = W_{t_{j}}^0,
\]

\[
W_{t_{j}}^0 = (1+r)\left[W_{t_{j}}^0 - p_j \phi^{(t_{j})}(W_{t_{j}}^0, p)\right] + Y_{t+1},
\]

\[
W_{t+2}^{t_{j+1}} = (1+r)\left[W_{t+2}^{t_{j+1}} - p_{t+1} \phi^{(t_{j+1})}(W_{t+1}^{t_{j+1}}, p)\right] + Y_{t+2},
\]

if preference is \( u^{t_{j+1}}(x_{t_{j+1}}) \) in period \( \ell+1 \);

\[
W_{t+3}^{t_{j+2}} = (1+r)\left[W_{t+3}^{t_{j+2}} - p_{t+2} \phi^{(t_{j+2})}(W_{t+2}^{t_{j+2}}, p)\right] + Y_{t+3},
\]

if preference is \( u^{t_{j+2}}(x_{t_{j+2}}) \) in period \( \ell+2 \);

\[
W_{t+4}^{t_{j+3}} = (1+r)\left[W_{t+4}^{t_{j+3}} - p_{t+3} \phi^{(t_{j+3})}(W_{t+3}^{t_{j+3}}, p)\right] + Y_{t+4},
\]

\[
W_{T}^{t_{j+T-1}} = (1+r)\left[W_{T}^{t_{j+T-1}} - p_{T} \phi^{(T)}(W_{T}^{t_{j+T-1}}, p)\right] + Y_{T};
\]

if preference is \( u^{T}(x_{T}) \) in period \( T-1 \);

\[
W_{T+1}^{t_{j+T}} = (1+r)\left[W_{T+1}^{t_{j+T}} - p_{T+1} \phi^{(T+1)}(W_{T+1}^{t_{j+T}}, p)\right] + Y_{T+1} = 0.
\]

**Problem C6.**

The inter-temporal Roy’s identity under stochastic life-span and preferences is derived from the consumer problem in which the consumer’s life-span involves \( \hat{T} \) periods where \( \hat{T} \) is a random variable with range \( \{1,2,\cdots,T\} \) and corresponding
probabilities \( \{ \gamma_1, \gamma_2, \cdots, \gamma_T \} \). Conditional upon the reaching of period \( \tau \), the probability of the consumer’s life-span would last up to periods \( \tau, \tau + 1, \cdots, T \) becomes respectively:

\[
\gamma_{\tau}, \gamma_{\tau+1}, \cdots, \gamma_{T}.
\]

The consumer maximizes his expected inter-temporal utility

\[
E_{\theta_1, \theta_2, \cdots, \theta_T} \left\{ \sum_{T=1}^{T} \gamma_T \sum_{k=1}^{T} \sum_{\nu_k=1}^{\tilde{m}_k} \rho_{\nu_k}^{h} \delta^k u^{k(\nu_k)}(x_k) \right\}
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{k+1} = W_k - \sum_{h=1}^{n_h} p_{h,k} x_{h,k} + r(W_k - \sum_{h=1}^{n_h} p_{h,k} x_{h,k}^h) + Y_{k+1}, \quad W_1 = W_0.
\]


C7. Inter-temporal Roy’s Identity under Stochastic Income and Preferences

\[
\frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial p_i^j} + \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\varphi^{(\nu_k)}(W_0^p, p), \quad \text{for } j \in \{1, 2, \cdots, n_i\};
\]

\[
\frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial p_h^k} + \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]

\[
\times \frac{\partial v^{(\nu_k)}(W_0^p, p)}{\partial W_0^p} = -\sum_{j=1}^{m_j} \sum_{j+1}^{m_{j+1}} \cdots \sum_{j+1}^{m_{j+2}} \cdots \sum_{h}^{m_h} \sum_{s=1}^{m_s} \sum_{s+1}^{m_{s+1}} \cdots \sum_{h+k}^{m_{h+k}} \cdots \sum_{h+k}^{m_{h+k}}
\]

\[
\times \varphi^{(\nu_k)}(W_0^p, p), \quad (1 + r)^{-(h-k)}
\]
or in an alternative form:
\[
\frac{\partial v^{(\ell+1)}(W^0, \rho)}{\partial \rho_h} = \sum_{j=1}^{m_{j+1}} \sum_{h=1}^{m_h} \sum_{\ell=1}^{m_{\ell+1}} \sum_{k=1}^{m_k} \rho^h \phi_h(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}, \rho), p) \phi_h(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}, \rho), p);
\]
for \( \ell \in \{1, 2, \ldots, T\} \), \( h \in \{\ell + 1, \ell + 2, \ldots, T\} \), \( k \in \{1, 2, \ldots, n_k\} \) and \( \nu \in \{1, 2, \ldots, m\} \), where
\[
W^\ell = W^\ell^0,
\]
\[
W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) = (1 + r[W^0 - p\phi^{(\nu_k)}(W^0, \rho)] + \theta^{j+1},
\]
\[
W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) = (1 + r[W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) - p\phi^{(\nu_k)}(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}), \rho)] + \theta^{j+1},
\]
if preference is \( \ell^{(\nu_{k-1})} \) in period \( \ell + 1 \);
\[
W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) = (1 + r[W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) - p\phi^{(\nu_k)}(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}), \rho)] + \theta^{j+1},
\]
if preference is \( \ell^{(\nu_{k-1})} \) in period \( \ell + 2 \);
\[
W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) = (1 + r[W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) - p\phi^{(\nu_k)}(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}), \rho)] + \theta^{j+1},
\]
if preference is \( \ell^{(\nu_{k-1})} \) in period \( T - 1 \);
\[
W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) = (1 + r[W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}) - p\phi^{(\nu_k)}(W^{j+1}(\theta^{j+1} \cdots \theta^{k+2}, \nu_k, \nu_{k-1} \cdots u_{-1}), \rho)] + \theta^{j+1} = 0.
\]

**Problem C7.**

The inter-temporal Roy’s identity under stochastic income and preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be \( u^{(\nu_k)}(x_k) \). His future preferences are not known with certainty. In particular, his utility function in period \( k \in \{2, 3, \ldots, T\} \) is known to be \( u^{(\nu_k)}(x_k) \) with probability \( \rho^{(\nu_k)} \) for \( \nu_k \in \{1, 2, \ldots, m\} \). We use \( \omega_k \) to denote the random variable with range \( \nu_k \in \{1, 2, \ldots, m\} \) and corresponding probabilities \( \{\rho^{(\nu_k)}, \rho^{(\nu_k)} \cdots, \rho^{(\nu_k)}\} \).

The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility
\[
E_{\omega_1, \omega_2, \ldots, \omega_T} \left\{ \sum_{k=1}^{T} \sum_{\nu_k=1}^{m} \rho^{(\nu_k)} \delta^{(\nu_k} u^{(\nu_k)}(x_k) \right\}
\]
\[
= E_{\omega_1, \omega_2, \ldots, \omega_T} \left\{ u^{(\nu_k)}(x_k) \right\}
\]
subject to the budget constraint characterized by the wealth dynamic
\[
W_{k+1} = W_k - \sum_{h=1}^{n} p^h W^h_k + r(W_k - \sum_{h=1}^{n} p^h W^h_k) + \theta_{k+1}, \quad W_1 = W^0_k.
\]
\( \theta_k \) is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \cdots, T\} \), is a set of statistically independent random variables, and \( E_{\theta, \theta, \cdots, \theta} \) is the expectation operation with respect to the statistics of \( \theta_2, \theta_1, \cdots, \theta_r \). The random variable \( \theta_k \) has a non-negative range \( \{ \theta^1_k, \theta^2_k, \cdots, \theta^m_k \} \) with corresponding probabilities \( \{ \lambda^1_k, \lambda^2_k, \cdots, \lambda^m_k \} \), for \( k \in \{2, \cdots, T\} \).


C8. Inter-temporal Roy’s Identity under Stochastic Income, Life-span and Preferences

\[ \frac{\partial V^{(it)}(W^0, \ell, p)}{\partial p^j} + \frac{\partial V^{(it)}(W^0, \ell, p)}{\partial W^0} \equiv -\phi_{\ell+j}(W^0, \ell, p), \text{ for } j \in \{1,2,\cdots,n_j\}; \]

\[ \frac{\partial V^{(it)}(W^0, \ell, p)}{\partial p^h} + \frac{\partial V^{(it)}(W^0, \ell, p)}{\partial W^0} \equiv -\sum_{j_{i+1}=1}^{m_{i+1}} \lambda^1_{j_{i+1}} - \sum_{j_{i+1}=1}^{m_{i+1}} \lambda^2_{j_{i+1}} - \cdots - \lambda^n_{j_{i+1}} - \sum_{j_{i+1}=1}^{m_{i+1}} \rho^1_{j_{i+1}} - \sum_{j_{i+1}=1}^{m_{i+1}} \rho^2_{j_{i+1}} - \cdots - \sum_{j_{i+1}=1}^{m_{i+1}} \rho^n_{j_{i+1}} \]

\[ \cdots \sum_{\nu_{i+1}=1}^{m_{i+1}} \rho^h_{\nu_{i+1}} \quad \delta h \quad \frac{\partial V^{(h)}(W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}, \ell, p)}{\partial W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}} \]

\[ (1 + r)^{-(h-\ell)} \]

\[ = \left[ \sum_{\kappa_{i+1}=1}^{m_{i+1}} \lambda^1_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \lambda^2_{\kappa_{i+1}} - \cdots - \lambda^n_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^1_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^2_{\kappa_{i+1}} - \cdots - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^n_{\kappa_{i+1}} \right] \]

\[ \delta h \quad \frac{\partial V^{(h)}(W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}, \ell, p)}{\partial W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}} \]

or in an alternative form:

\[ \frac{\partial V^{(it)}(W^0, \ell, p)}{\partial p^h} \equiv -\sum_{\kappa_{i+1}=1}^{m_{i+1}} \lambda^1_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \lambda^2_{\kappa_{i+1}} - \cdots - \lambda^n_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^1_{\kappa_{i+1}} - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^2_{\kappa_{i+1}} - \cdots - \sum_{\kappa_{i+1}=1}^{m_{i+1}} \rho^n_{\kappa_{i+1}} \]

\[ \delta h \quad \frac{\partial V^{(h)}(W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}, \ell, p)}{\partial W^0_{\theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}, \cdots, \theta^h_{i+1}}} \]

for \( \ell \in \{1,2,\cdots,T\}, \ h \in \{\ell+1,\ell+2,\cdots,T\}, \ k \in \{1,2,\cdots,n_h\} \) and \( \nu_i \in \{1,2,\cdots,m_i\} \), where

\[ W^0_{\ell} = W^0, \]

\[ W^0_{\ell+1,\ell+2} = (1 + r)[W^0_{\ell+1} - p_{\ell+2,\ell+1}(W^0_{\ell+1}, \ell+2)] + \theta^h_{\ell+1}, \]

\[ W^0_{\ell+1,\ell+2} = (1 + r)[W^0_{\ell+1} - p_{\ell+2,\ell+1}(W^0_{\ell+1, \ell+2}, \ell+2)] + \theta^h_{\ell+2}, \]

if preference is \( u^{(\ell+1)}(x_{\ell+1}) \) in period \( \ell + 1 \).
subject to the budget

$$\sum_{\ell=0}^{\tau} p_{\ell} x^{\ell} = W_{\ell}, \quad \ell = 0, 1, \ldots, T.$$
The demand function of commodity $h$ in period $t$ is
\[ \phi^h_t(W^0_t, p_t, p_{t+1}, \ldots, p_T) \]
subject to the budget constraint characterized by the wealth dynamics
\[ W_{k+1} = W_k - \sum_{h=1}^{n_k} p^h_k x^h_k + r(W_k - \sum_{h=1}^{n_k} p^h_k x^h_k) + Y_{k+1}, \quad W_1 = W^0_1, \]
where
\[ x_k = (x^1_k, x^2_k, \ldots, x^{n_k}_k) \]
is the vector of quantities of goods consumed in period $k$, $p_k = (p^1_k, p^2_k, \ldots, p^{n_k}_k)$ is price vector, $r$ is the interest rate, $Y_k$ is the income that the consumer will receive in period $k$, $\delta^k = \left( \prod_{c=2}^{k} \beta_{c} \right)$ is the discount factor with $\beta_{c}$ being the consumer's subjective one-period discount factor for the duration from period $\tau - 1$ to period $\tau$, $\beta_1 = 1$ for the discount factor in the initial period 1 and $\delta^k = \left( \prod_{c=1}^{k} \beta_{c} \right)$ is the discount factor for the duration from period $\tau - 1$ to period $\tau$. The period $k$ utility function $u^k(x^1_k, x^2_k, \ldots, x^{n_k}_k)$ is continuously differentiable and quasi-concave yielding convex level (indifference) curves. The time preference factor is embodied in the utility function. The time preference factor is embodied in the utility function. The amount of unconsumed wealth $W_k - p_k x_k$ in period $k$ will generate an interest income $r(W_k - p_k x_k)$ in period $k + 1$. 

In addition $\phi^h_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)$ is the ordinary demand function of commodity $h$ in period $\ell$, and $\psi^h_\ell(W^0_\ell, p_\ell, p_{\ell+1}, \ldots, p_T)$ is the wealth compensated demand function of commodity $h$ in period $\ell$.
C10. Dynamic Slutsky Equation under Stochastic Income

\[
\frac{\partial \nu^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial p^j_t} = \frac{\partial \psi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial p^j_t} - \frac{\partial \psi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial W^0_t} \varphi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T),
\]

for \( j \in \{1, 2, \ldots, n_k\} \), and

\[
\frac{\partial \nu^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial p^j_t} = \frac{\partial \psi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial p^j_t} - \frac{\partial \psi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)}{\partial W^0_t} \varphi^j_t(W^0_t, p_t, p_{t+1}, \ldots, p_T)
\]

\[
\times \sum_{w_{j+1}=1}^{m_{j+1}} \sum_{w_{j+2}=1}^{m_{j+2}} \cdots \sum_{w_{k}=1}^{m_{k}} \delta_{t+1}^k \frac{\partial v^k(W^0_t, p, p_{t+1}, \ldots, p_T)}{\partial W^0_t} \varphi^j_t(W^0_t, p, p_{t+1}, \ldots, p_T) (1 + r)^{-(k-\ell)},
\]

for \( \ell \in \{1, 2, \ldots, T - 1\}, k \in \{k + 1, k + 2, \ldots, T\} \) and \( j \in \{1, 2, \ldots, n_k\} \).

Problem C10.
The dynamic Slutsky equation under stochastic income is derived from the consumer problem in which the consumer maximizes his expected inter-temporal utility

\[
E_{\theta_2, \theta_3, \ldots, \theta_T} \left\{ \sum_{k=1}^T \delta_t^k u^k(x^1_k, x^2_k, \ldots, x^n_k) \right\} = E_{\theta_2, \theta_3, \ldots, \theta_T} \left\{ \sum_{k=1}^T \delta_t^k u^k(x_k) \right\}
\]

subject to the budget constraint characterized by the stochastic wealth dynamics

\[
W_{k+1} = (1 + r)(W_k - p_k x_k) + \theta_{k+1}, \quad W_1 = W^0_1,
\]

where \( \theta_k \) is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \ldots, T\} \), is a set of statistically independent random variables, and \( E_{\theta_2, \theta_3, \ldots, \theta_T} \) is the expectation operation with respect to the statistics of \( \theta_2, \theta_3, \ldots, \theta_T \). The random variable \( \theta_k \) has a non-negative range \( \{\theta^1_k, \theta^2_k, \ldots, \theta^n_k\} \) with corresponding probabilities \( \{\lambda^1_k, \lambda^2_k, \ldots, \lambda^n_k\} \), for \( k \in \{2, \ldots, T\} \).

C11. Dynamic Slutsky Equation under Stochastic Life-span

\[
\frac{\partial \phi^k(W^0, p, p_{t+1}, \ldots, p_T)}{\partial p^1_k} = \frac{\partial \psi^h(W^0, p, p_{t+1}, \ldots, p_T)}{\partial p^1_k} - \frac{\partial \phi^h(W^0, p, p_{t+1}, \ldots, p_T)}{\partial W^0_t} (1 + r)^{-k} \phi^j(W^0_k, p_k, p_{k+1}, \ldots, p_T),
\]

for \( j \in \{1, 2, \ldots, n_k\} \) and \( k \in \{\ell, \ell + 1, \ldots, T\} \).

Problem C11.
The dynamic Slutsky equation under stochastic life-span is derived from the consumer problem in which the consumer’s life-span involves \( \hat{T} \) periods where \( \hat{T} \) is a random variable with range \( \{1, 2, \ldots, T\} \) and corresponding probabilities \( \{\gamma_1, \gamma_2, \ldots, \gamma_T\} \). Conditional upon the reaching of period \( \tau \), the probability of the consumer’s life-span would last up to periods \( \tau, \tau + 1, \ldots, T \) becomes respectively

\[
\frac{\gamma_T}{\sum_{\zeta=\tau}^T \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^T \gamma_\zeta}, \ldots, \frac{\gamma_T}{\sum_{\zeta=\tau}^T \gamma_\zeta}.
\]

The consumer maximizes his expected inter-temporal utility

\[
\sum_{t=1}^T \gamma_T \sum_{k=1}^t \delta^k u^k(x_k),
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{k+1} = W_k - \sum_{h=1}^{n_k} p_h x_h^k + r(W_k - \sum_{h=1}^{n_k} p_h x_h^k) + Y_{k+1}, \quad W_1 = W^0_1.
\]

where

\( r \) is the interest rate, \( Y_h \) is the income that the consumer will receive in period \( k \).

for $j \in \{1, 2, \cdots, n_j\}$, and
\[
\frac{\partial \phi_j^h(W_j^0, p_j, p_{j+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial \psi_j^h(V_j^0, p_j, p_{j+1}, \cdots, p_T)}{\partial W_j^0} \sum_{m_{j+1}}^{m_j} \sum_{m_{j+2}}^{m_{j-1}} \sum_{m_{j+1}}^{m_{j-1}} \lambda_{j+1}^k \lambda_{j+2}^k \cdots \lambda_{j+1}^k
\]
\[
\frac{\partial \phi_j^h(W_j^0, p_j, p_{j+1}, \cdots, p_T)}{\partial W_j^0} \sum_{m_{j+1}}^{m_j} \sum_{m_{j+2}}^{m_{j-1}} \sum_{m_{j+1}}^{m_{j-1}} \lambda_{j+1}^k \lambda_{j+2}^k \cdots \lambda_{j+1}^k \delta_k^j \frac{\partial v^k(W_k^{0, j1, j2, \cdots, j8}, p)}{\partial W_k^{0, j1, j2, \cdots, j8}}
\]
\[
\times \frac{\partial v^k(W_k^{0, j1, j2, \cdots, j8}, p)}{\partial W_k^{0, j1, j2, \cdots, j8}} \phi_j^h(W_k^{0, j1, j2, \cdots, j8}, p) (1 + r)^{-k-\ell},
\]
for $\ell \in \{1, 2, \cdots, T - 1\}, k \in \{k + 1, k + 2, \cdots, T\}$ and $j \in \{1, 2, \cdots, n_j\}$.

**Problem C12.**

The dynamic Slutsky equation under stochastic income and life-span is derived from the consumer problem in which the consumer’s life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1, 2, \cdots, T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period $\tau$, the probability of the consumer’s life-span would last up to periods $\tau, \tau + 1, \cdots, T$ becomes respectively

\[
\sum_{\tau = \tau}^{T} \gamma_{\tau} = \sum_{\tau = \tau}^{T} \gamma_{\tau + 1} = \sum_{\tau = \tau}^{T} \gamma_{\tau + 2} = \cdots = \gamma_{\tau + T}.
\]

The consumer maximizes his expected inter-temporal utility

\[
E_{\theta_2, \theta_3, \cdots, \theta_T} \left\{ \sum_{T = 1}^{T} \sum_{k = 1}^{T} \delta_t^k u^k(x_t) \right\},
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h) + \theta_{k+1}, \quad W_1 = W_1^0.
\]

where $\theta_k$ is the random income that the consumer will receive in period $k$; and $\theta_k$, for $k \in \{2, \cdots, T\}$, is a set of statistically independent random variables, and $E_{\theta_2, \theta_3, \cdots, \theta_T}$ is the expectation operation with respect to the statistics of $\theta_2, \theta_3, \cdots, \theta_T$. The random variable $\theta_k$ has a non-negative range $\{\theta_1^k, \theta_2^k, \cdots, \theta_m^k\}$ with corresponding probabilities $\{\hat{\lambda}_1^k, \hat{\lambda}_2^k, \cdots, \hat{\lambda}_m^k\}$, for $k \in \{2, \cdots, T\}$.

**References:**


C13. Dynamic Slutsky Equation under Stochastic Preferences

\[
\frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} = \frac{\partial \psi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} - \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \varphi_{i}(W_{\ell}^0, p) ,
\]

\[
\frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} = \frac{\partial \psi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} - \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1} \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1}
\]

\[
\cdots \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1} \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1}
\]

\[
\frac{\partial \nu^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} (1 + r)^{(-k)}
\]

\[
\frac{\partial \nu^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} (1 + r)^{(-k)}
\]

\[
\left[ \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1} \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1} \right]
\]

\[
\text{for } \ell \in \{1, 2, \cdots, T\}, k \in \{k+1, k+2, \cdots, T\}, i_k \in \{1, 2, \cdots, n_k\}, h, i, \{1, 2, \cdots, n_k\} \text{ and } \nu_{\ell} \in \{1, 2, \cdots, \nu_{\ell}\}.
\]

Problem C13.

The dynamic Slutsky equation under stochastic preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be \(u^{(1)}(x_i)\). His future preferences are not known with certainty. In particular, his utility function in period \(k \in \{2, 3, \cdots, T\}\) is known to be \(u^{(k)(v)}(x_i)\) with probability \(\rho^{\nu}_k\) for \(\nu_k \in \{1, 2, \cdots, \nu_{\ell}\}\). We use \(\tilde{\nu}_k\) to denote the random variable with range \(\nu_k \in \{1, 2, \cdots, \nu_{\ell}\}\) and corresponding probabilities \(\{\rho^{\nu}_1, \rho^{\nu}_2, \cdots, \rho^{\nu}_{\nu_{\ell}}\}\). The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

\[
E_{\theta, \theta, \cdots, \theta} \left\{ \sum_{k=1}^{T} \sum_{\nu_k=1}^{m_k} \rho^{\nu}_k \delta^{\nu}(\nu_k) u^{(k)(\nu_k)}(x_i) \right\}
\]

\[
= E_{\theta, \theta, \cdots, \theta} \left\{ u^{(1)}(x_i) + \sum_{k=2}^{T} \sum_{\nu_k=1}^{m_k} \rho^{\nu}_k \delta^{\nu}(\nu_k) u^{(k)(\nu_k)}(x_i) \right\}
\]

subject to the budget constraint characterized by the wealth dynamic

\[
W_{k+1} = W_k - \sum_{h=1}^{l_k} \rho^{h}_k x_k^h + r(W_k - \sum_{h=1}^{l_k} \rho^{h}_k x_k^h) + y_{k+1}, \quad W_1 = W_1^0.
\]


C14. Dynamic Slutsky Equation under Stochastic Life-span and Preferences

\[
\frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} = \frac{\partial \psi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} - \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \varphi_{i}(W_{\ell}^0, p) ,
\]

\[
\frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} = \frac{\partial \psi^{(h)}(W_{\ell}^0, p)}{\partial \rho^{i\rho}_{i}} - \frac{\partial \phi^{(h)}(W_{\ell}^0, p)}{\partial W_{\ell}^0} \sum_{i_{\tau+1} = 1}^{m_{\tau+1}} \sum_{i_{\tau-2} = 1}^{m_{\tau-2}} \cdots \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1} \rho^{i_{\tau+1}i_{\tau-2} \cdots i_1}
\]
Problem C14.
The dynamic Slutsky equation under stochastic life-span and preferences is derived from the consumer problem in which the consumer’s life-span involves $\tilde{T}$ periods where $\tilde{T}$ is a random variable with range $[1,2,\ldots,T]$ and corresponding probabilities $\{\gamma_1, \gamma_2, \ldots, \gamma_T\}$. Conditional upon the reaching of period $\tau$, the probability of the consumer’s life-span would last up to periods $\tau, \tau+1, \ldots, T$ becomes respectively:

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \ldots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}.$$  

The preference or utility function of the consumer in period 1 is known to be $u^{(1)}(x_1)$. His future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\ldots,T\}$ is known to be $u^{(x_k)}(x_k)$ with probability $\rho_k^{x_k}$ for $\nu_k \in \{1,2,\ldots,\bar{m}_k\}$ if he survives in period $k$. We use $\tilde{\nu}_k$ to denote the random variable with range $\nu_k \in \{1,2,\ldots,\bar{m}_k\}$ and corresponding probabilities $\{\rho_1^{x_k}, \rho_2^{x_k}, \ldots, \rho_{\bar{m}_k}^{x_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{T=1}^{T} \gamma_T \sum_{k=1}^{\tilde{T}} \sum_{\nu_k=1}^{\bar{m}_k} \rho_k^{x_k} \delta^k u^{(x_k)}(x_k) \right\}$$

$$= E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ u^{(1)}(x_1) + \sum_{T=2}^{T} \gamma_T \sum_{k=1}^{\tilde{T}} \sum_{\nu_k=1}^{\bar{m}_k} \rho_k^{x_k} \delta^k u^{(x_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{\bar{m}_k} p_k^{h} x_k^{h} + r(W_k - \sum_{h=1}^{\bar{m}_k} p_k^{h} x_k^{h}) + Y_{k+1}, \quad W_1 = W_0^0.$$


C15. Dynamic Slutsky Equation under Stochastic Income and Preferences

$$\frac{\partial \phi_i^{(x_k)}(W_0^{0}, p)}{\partial p_i^{x_k}} = \frac{\partial \psi_i^{(x_k)}(W_0^{0}, p)}{\partial p_i^{x_k}} - \frac{\partial \phi_i^{(x_k)}(W_0^{0}, p)}{\partial W_0^{0}} \phi'_i(W_0^{0}, p),$$
\[
\frac{\partial \varphi^{(x_k)}(W_0^0, p)}{\partial p^i_k} = \frac{\partial \varphi^{(x_k)}(W_0^0, p)}{\partial W_0^0}
\times \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \cdots \sum_{j_{\ell-1}=1}^{m_{\ell-1}} \sum_{j_{\ell}=1}^{m_{\ell}} \sum_{i_{\ell+1}=1}^{m_{\ell+1}} u_{i_{\ell+1}} \sum_{i_{\ell+2}=1}^{m_{\ell+2}} \cdots \\
\cdots \sum_{i_{m-1}=1}^{m_{m-1}} \rho^i_k \delta^i_{l+1} \frac{\partial \varphi^{(x_k)}(W_{\ell}^{(x_k)}(\theta_1^{(x_k)}, \ldots, \theta_{\ell-1}^{(x_k)}), p)}{\partial W_{\ell}^{(x_k)}(\theta_1^{(x_k)}, \ldots, \theta_{\ell-1}^{(x_k)}), p}
\]

(1 + r)^{-(\ell-k)}

+ \left[ \sum_{m_{\ell+1}=1}^{m_{\ell+1}} \sum_{\ell_{\ell+1}=1}^{m_{\ell+1}} \cdots \sum_{j_{\ell-1}=1}^{m_{\ell-1}} \sum_{j_{\ell}=1}^{m_{\ell}} \sum_{i_{\ell+1}=1}^{m_{\ell+1}} \rho_{\ell+1} \delta^i_{l+1} \frac{\partial \varphi^{(x_k)}(W_{\ell+1}^{(x_k)}(\theta_1^{(x_k)}, \ldots, \theta_{\ell}^{(x_k)}), p)}{\partial W_{\ell+1}^{(x_k)}(\theta_1^{(x_k)}, \ldots, \theta_{\ell}^{(x_k)}), p} \right],

for \( \ell \in \{1, 2, \ldots, T\}, k \in \{\ell + 1, \ell + 2, \ldots, T\}, i_k \in \{1, 2, \ldots, m_k\}, h, i_k \in \{1, 2, \ldots, n_k\} \) and

\( \nu_k \in \{1, 2, \ldots, m_i\} \).

\[\text{Problem C15.}\]

The dynamic Slutsky equation under stochastic income and preferences is derived from the consumer problem in which the preference or utility function of the consumer in period 1 is known to be \( u^{(x_1)}(x_1) \). His future preferences are not known with certainty. In particular, his utility function in period \( k \in \{2, 3, \ldots, T\} \) is known to be \( u^{(x_k)}(x_k) \) with probability \( \rho_k^{(x_k)} \) for \( \nu_k \in \{1, 2, \ldots, m_k\} \). We use \( \tilde{\nu}_k \) to denote the random variable with range \( \nu_k \in \{1, 2, \ldots, m_k\} \) and corresponding probabilities \( \{\rho_k^1, \rho_k^2, \ldots, \rho_k^{m_k}\} \).

The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

\[ E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ \sum_{j=1}^{T} \sum_{i_{j+1}=1}^{n_{j+1}} \rho_{j+1}^{(x_k)} \delta_{j+1}^{(x_k)} (x_{j+1}) \right\} \]

\[ = E_{\theta_1, \theta_2, \ldots, \theta_T} \left\{ u^{(x_1)}(x_1) + \sum_{j=2}^{T} \sum_{i_{j+1}=1}^{n_{j+1}} \rho_{j+1}^{(x_k)} \delta_{j+1}^{(x_k)} (x_{j+1}) \right\} \]

subject to the budget constraint characterized by the wealth dynamic

\[ W_{k+1} = W_k - \sum_{h=1}^{n} p_k^h x_h + r(W_k - \sum_{h=1}^{n} p_k^h x_h) + \theta_{k+1}, \quad W_1 = W_0^0. \]

where \( \theta_k \) is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \ldots, T\} \), is a set of statistically independent random variables, and \( E_{\theta_1, \theta_2, \ldots, \theta_T} \) is the expectation operation with respect to the statistics of \( \theta_1, \theta_2, \ldots, \theta_T \). The random variable \( \theta_k \) has a non-negative range \( \{\theta_k^1, \theta_k^2, \ldots, \theta_k^{m_k}\} \) with corresponding probabilities \( \{\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^{m_k}\} \), for \( k \in \{2, \ldots, T\} \).

C16. Dynamic Slutsky Equation under Stochastic Income, Life-span and Preferences

\[
\frac{\partial \varphi_{i}^{(u)}(W_0^0, p)}{\partial p_i} = \frac{\partial \psi_{i}^{(u)}(W_0^0, p)}{\partial p_i} - \frac{\partial \varphi_{i}^{(u)}(W_0^0, p)}{\partial W_0^0},
\]

\[
\frac{\partial \varphi_{k}^{(u)}(W_0^0, p)}{\partial p_k} = \frac{\partial \psi_{k}^{(u)}(W_0^0, p)}{\partial p_k} - \frac{\partial \varphi_{k}^{(u)}(W_0^0, p)}{\partial W_0^0}.
\]

\[
\times \sum \frac{\rho_{k}^{(u)}}{\delta_{i+1}} \left( \frac{\partial \nu_{i+k}}{\partial W_{k}^{(u)}} \right), p \varphi_{k}^{(u)}(W_{k}^{(u)}, p)
\]

\[
(1 + r)^{\ell-k}
\]

\[
+ \left[ \sum_{w_{i+1}}^{m_{i+1}} \sum_{w_{i+2}}^{m_{i+2}} \sum_{w_{i+3}}^{m_{i+3}} \sum_{w_{i+4}}^{m_{i+4}} \rho_{i+1}^{(u)} \sum_{i+2}^{m_{i+2}} \rho_{i+2}^{(u)} \sum_{i+3}^{m_{i+3}} \rho_{i+3}^{(u)} \sum_{i+4}^{m_{i+4}} \rho_{i+4}^{(u)} \right]
\]

for \( \ell \in \{1,2,\cdots,T\} \), \( k \in \{\ell+1,\ell+2,\cdots,T\} \), \( i_k \in \{1,2,\cdots,n_k\} \), \( h \in \{1,2,\cdots,n_i\} \) and \( \nu_k \in \{1,2,\cdots,m_k\} \).

Problem C16.
The dynamic Slutsky equation under stochastic income, life-span and preferences is derived from the consumer problem in which the consumer’s life-span involves \( \hat{T} \) periods where \( \hat{T} \) is a random variable with range \( \{1,2,\cdots,T\} \) and corresponding probabilities \( \{\gamma_1,\gamma_2,\cdots,\gamma_T\} \). Conditional upon the reaching of period \( \tau \), the probability of the consumer’s life-span would last up to periods \( \tau,\tau+1,\cdots,T \) becomes respectively:

\[
\frac{\gamma_{\tau}}{\sum_{\zeta=\tau}^{\gamma_{\zeta}}} , \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{\gamma_{\zeta}}} , \cdots , \frac{\gamma_{\tau}}{\sum_{\zeta=\tau}^{\gamma_{\zeta}}} .
\]

The preference or utility function of the consumer in period 1 is known to be \( u_{1}^{(v)}(x_1) \). His future preferences are not known with certainty. In particular, his utility function in period \( k \in \{2,3,\cdots,T\} \) is known to be \( u_{k}^{(v_k)}(x_k) \) with probability \( \rho_{k}^{(v_k)} \) for \( \nu_k \in \{1,2,\cdots,m_k\} \) if he survives in period \( k \). We use \( \hat{\nu}_k \) to denote the random variable.
with range \( \nu_k \in \{1, 2, \cdots, \bar{m}_k \} \) and corresponding probabilities \( \{p_k^1, p_k^2, \cdots, p_k^m \} \). The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

\[
E_{\theta_k, \theta_{k-1}, \cdots, \theta_1} \left\{ \sum_{T=1}^{T} \gamma_T \sum_{k=1}^{T} \sum_{\nu_k=1}^{m_k} \rho_k^{(k)} \delta_k^{(k)} u^{(k)}(x_k) \right\} = E_{\theta_k, \theta_{k-1}, \cdots, \theta_1} \left\{ u^{(1)}(x_1) + \sum_{T=2}^{T} \gamma_T \sum_{k=2}^{T} \sum_{\nu_k=1}^{m_k} \rho_k^{(k)} \delta_k^{(k)} u^{(k)}(x_k) \right\},
\]

subject to the budget constraint characterized by the wealth dynamics

\[
W_{k+1} = W_k - \sum_{h=1}^{m_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{m_k} p_k^h x_k^h) + \theta_{k+1}, \quad W_1 = W_1^0.
\]

where \( \theta_k \) is the random income that the consumer will receive in period \( k \); and \( \theta_k \), for \( k \in \{2, \cdots, T\} \), is a set of statistically independent random variables, and \( E_{\theta_k, \theta_{k-1}, \cdots, \theta_1} \) is the expectation operation with respect to the statistics of \( \theta_1, \theta_1, \cdots, \theta_T \). The random variable \( \theta_k \) has a non-negative range \( \{\theta_k^1, \theta_k^2, \cdots, \theta_k^m\} \) with corresponding probabilities \( \{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^m\} \), for \( k \in \{2, \cdots, T\} \).


D. Biological Population Density Functions

D1. Stationary Density Function of Generalized Stochastic Food-chain of the Lotka-Volterra-Yeung Type

The function

\[
\psi(N) = m \prod_{i=1}^{n} \frac{1}{N_i} \exp \left[ 2A_i \ln N_i - 2F_i(\ln N_i) + 2F_i(0) \right]/\sigma^2
\]

gives the stationary probability density of species \( N_1, N_2, \cdots, N_n \) of the generalized Lotka-Volterra-Yeung type of stochastic food-chain:

\[
dN_1(t) = [\alpha_1 N_1(t) - b_1 N_1(t) f_1(N_1(t)) - \nu_1 N_1(t) f_2(N_2(t))] dt + \sigma \sqrt{b_1} N_1(t) dz(t),
\]

\[
dN_2(t) = [\alpha_2 N_2(t) - b_2 N_2(t) f_2(N_2(t)) - \nu_2 N_2(t) f_3(N_3(t)) + \nu_1 N_1(t) f_1(N_1(t))] dt + \sigma \sqrt{b_2} N_2(t) dz(t),
\]

\[
dN_3(t) = [\alpha_3 N_3(t) - b_3 N_3(t) f_3(N_3(t)) - \nu_3 N_3(t) f_4(N_4(t)) + \nu_2 N_2(t) f_2(N_2(t))] dt + \sigma \sqrt{b_3} N_3(t) dz(t),
\]

\[\vdots\]

\[
dN_{n-1}(t) = [\alpha_{n-1} N_{n-1}(t) - b_{n-1} N_{n-1}(t) f_{n-1}(N_{n-1}(t)) - \nu_{n-1} N_{n-1}(t) f_n(N_n(t)) + \nu_{n-2} N_{n-2}(t) f_{n-2}(N_{n-2}(t))] dt + \sigma \sqrt{b_{n-1}} N_{n-1}(t) dz(t),
\]

\[
dN_n(t) = [\alpha_n N_n(t) - b_n N_n(t) f_n(N_n(t)) + \nu_{n-1} N_{n-1}(t) f_{n-1}(N_{n-1}(t))] dt
\]

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\[
+ \sigma \sqrt{b_n} \dot{N}_n(t) dz(t),
\]
where \( N_i(t) \) is the population level of the species in the \( i \)th trophic level at time \( t \);
\( v_i \) for \( i \in \{1,2,\ldots,n-1\} \) are positive constants, \( b_i \) is positive and \( b_i \) for \( i \in \{2,3,\ldots,n\} \) are nonnegative constants;
\( \alpha_i > 0 \), and \( \alpha_i \) for \( i \in \{2,3,\ldots,n\} \) are constants with \( \alpha_i \) being positive when \( b_i > 0 \)
and negative when \( b_i = 0 \);
\( f_i(0) = 0 \) and \( f_i(N_i) > 0 \) for positive values of \( N_i \), and \( f_i(N_i) \) is a continuous differentiable and monotonically increasing in \( N_i \), and \( f_i(e^s) \) is an integrable function yielding \( \int_0^t f_i(e^s) ds = F_i(x_i) - F_i(0) \), for \( i = 1,2,\ldots,n \);
and
\[ A_1, A_2, \ldots, A_n \] satisfies
\[ b_1 A_1 + v_1 A_2 = \omega_1, \]
\[- v_1 A_1 + b_2 A_2 + v_2 A_3 = \omega_2, \]
\[- v_2 A_2 + b_3 A_3 + v_3 A_4 = \omega_3, \]
\[ \vdots \]
\[- v_{n-2} A_{n-2} + b_{n-1} A_{n-1} + v_{n-1} A_n = \omega_{n-1}, \]
\[- v_{n-1} A_{n-1} + b_n A_n = \omega_n. \]


Note:
Using the generalized density function D1 one can also obtain the stationary density function of the stochastic Lotka-Volterra food-chain in Yeung (1988):
\[
dN_1(t) = [a_1 N_1(t) - bN_1^2(t) - c_1 N_1(t)(N_2(t))] dt + \varepsilon N_1(t) dz(t),
\]
\[
dN_i(t) = [-a_i N_i(t) - c_i N_i(t)(N_{i+1}(t)) + \gamma_i N_i(t) N_{i-1}(t)] dt,
\]
for \( i = 2,3,\ldots,n-1, \)
\[
dN_n(t) = [-a_n N_n(t) + \gamma_n N_n(t) N_{n-1}(t)] dt,
\]
where \( N_i(t) \) is the population level of the species in the \( i \)th trophic level at time \( t \),
\( z(t) \) is a standard Wiener process, with \( E(dz_i) = 0, E(dz_i^2) = dt \) and \( E(dtdz_i) = 0, b \),
a_i for \( i \in \{1,2,3,\ldots,n\} \) and c_i for \( i \in \{1,2,3,\ldots,n-1\} \) and \( \gamma_i \) for \( i \in \{2,3,\ldots,n\} \) are positive constants, and \( \varepsilon \) is a constant.

Similarly, using the generalized density function D1 one can also obtain the stationary density function of the prey species \( N_1 \) and the predator species \( N_2 \) of the predator prey system in Yeung (1986):
\[
dN_1(t) = [a_1 N_1(t) - bN_1^2(t) - c_1 N_1(t)(N_2(t))] dt + \varepsilon N_1(t) dz(t),
\]
\[
dN_2(t) = [-a_2 N_2(t) + \gamma N_2(t) N_1(t)] dt,
\]

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where \(a_1, a_2, b, \gamma\) and \(\varepsilon\) are positive constants, and \(z(t)\) is a standard Wiener process, with \(E(dz_t) = 0, E(dz_t^2) = dt\).


E. Number Theory

E1. The Number of Embedded Coalitions

The number of embedded coalitions in a \(n\)-person game is:

\[
Y(1) = \sum_{t=0}^{n} \binom{1}{t} = \binom{1}{0} = 1, \quad \text{for } n=1;
\]

\[
Y(2) = \sum_{t=0}^{n} \binom{2}{t} = \binom{2}{1} + \binom{2}{0} = 3, \quad \text{for } n=2;
\]

\[
Y(n) = \sum_{t=1}^{n} \binom{n}{t} \left( \sum_{k=1}^{t-1} Y(k) \right) + \sum_{t=0}^{n-1} \binom{n-1}{t}, \quad \text{for } n \geq 3.
\]

Problem E1.
Let \(N = \{1,2,\ldots,n\}\) be a finite set of \(n\) players in a \(n\)–person game. The subsets of \(N\) are coalitions. A partition \(\Lambda\) is formed by disjoint non-empty subsets of \(N\) representing a way that these \(n\) players are joined. Given a partition \(\Lambda\) and a coalition \(S \subset N\), the pair \((S, \Lambda)\) is called an embedded coalition, that is the coalition \(S\) embedded in partition \(\Lambda\). The Bell (1934) number, denoted by \(\beta(n)\), gives the number of partitions in a \(n\)–person game. The number of embedded coalitions in a partition is the number of subsets formed in that partition. The total number of embedded coalitions \(Y(n)\) in a \(n\)–person game is the sum of the numbers of embedded coalitions in the \(\beta(n)\) partitions of \(N\).


E2. The Number of Embedded Coalitions where the position of the individual player counts

The number of embedded coalitions in a \(n\)-person game where the position of the individual player counts is:
\[
\varphi(1) = 1 \sum_{t=0}^{1} \binom{1}{t} = 1, \quad \text{for } n = 1;
\]

\[
\varphi(2) = 2 \sum_{t=0}^{2} \binom{2}{t} = 6, \quad \text{for } n = 2;
\]

\[
\varphi(n) = n! \left[ \sum_{t=2}^{n-1} \binom{n}{t} \left( \sum_{k=1}^{n-1} \varphi(k) \cdot \frac{1}{k!} \right) + \sum_{t=0}^{n-1} \binom{n}{t} \right], \quad \text{for } n \geq 3.
\]

**Problem E2.**
Consider the problem in Problem E1 in which the position of the individual player in a embedded coalition counts. The total number of embedded coalitions \(\varphi(n)\) in a is the sum of the numbers of embedded coalitions with the positions of individual players count in the \(\beta(n)\) partitions of \(N\).